Future King

Future king, the king of future stationary. NOTEBOOK

— PHYSICS

塞下秋来风景异
衡阳雁去无留意
四面边声连角起
千嶂里
长烟落日孤城闭

浊酒一杯家万里
燕然未勒归无计
羌管悠悠霜满地
人不寐
将军白发征夫泪

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刘昂之

一. 牛顿力学

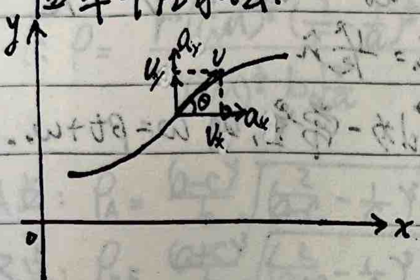
1. 运动、动力学

① 线量：$\vec{s}=\vec{r_2}-\vec{r_1}$，$\vec{v}=\frac{d\vec{s}}{dt}$，$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{s}}{dt^2}$

（自然系中 $\vec{a}=\frac{dv}{dt}\vec{\tau}+\frac{v^2}{\rho}\vec{n}$）

② 角量：$\theta=\varphi_2-\varphi_1$，$\omega=\frac{d\theta}{dt}$，$\beta=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$，$\vec{v}=\vec{\omega}\times\vec{\rho}$

曲率半径求法：



1) $\rho=\frac{ds}{d\theta}=\frac{v^3}{|\vec{a}\times\vec{v}|}=\frac{v^2}{a_n}$

2). $v=\frac{\sqrt{(dx)^2+(dy)^2}}{dt}$

$$a_n=a_x\sin\theta-a_y\cos\theta=\frac{d^2x}{dt^2}\frac{\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}}-\frac{d^2y}{dt^2}\frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$=\frac{d(\frac{dx}{dt})}{dt}\frac{dy}{\sqrt{(dx)^2+(dy)^2}}-\frac{d(\frac{dy}{dt})}{dt}\frac{dx}{\sqrt{(dx)^2+(dy)^2}}$$

$$\therefore \rho=\frac{v^2}{a_n}=\frac{\frac{(dx)^2+(dy)^2}{(dt)^2}}{\frac{1}{dt\sqrt{(dx)^2+(dy)^2}}\left[d(\frac{dx}{dt})dy-d(\frac{dy}{dx}\cdot\frac{dx}{dt})dx\right]}$$

$$=\frac{[(dx)^2+(dy)^2]^{\frac{3}{2}}}{dt\left[d(\frac{dx}{dt})dy-\frac{dy}{dx}\cdot d(\frac{dx}{dt})dx-\frac{dx}{dt}d(\frac{dy}{dx})dx\right]}$$

$$=-\frac{[(dx)^2+(dy)^2]^{\frac{3}{2}}}{d(\frac{dy}{dx})(dx)^2}$$

对于椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，有

$$\rho=\frac{\left[(dx)^2+\frac{b^2}{a^2}\frac{x^2(dx)^2}{a^2-x^2}\right]^{\frac{3}{2}}}{\frac{\frac{b}{a}(\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}})dx\,(dx)^2}{a^2-x^2}}=\frac{(1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2})^{\frac{3}{2}}}{\frac{b}{a}\cdot\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}=\frac{(a^4-c^2x^2)^{\frac{3}{2}}}{a^4b}$$

$$\begin{cases} x=0\text{时},\ \rho=\frac{a^2}{b} \\ x=a\text{时},\ \rho=\frac{b^2}{a} \end{cases}$$

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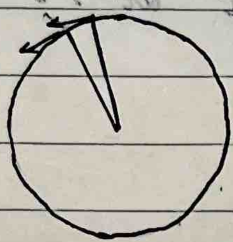
③ 典例：

1) 匀速直线：$\vec{r}=\vec{v}t$

2) 匀加速直线：$\vec{v}=\vec{a}t+\vec{v_0}$，$\vec{r}=\int\vec{v}\,dt=\vec{r_0}+\vec{v_0}t+\frac{1}{2}\vec{a}t^2$.

3) 圆周

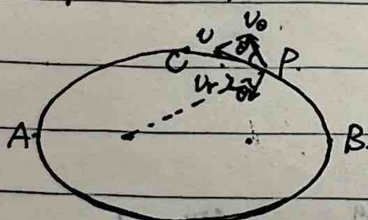


$\vec{a_n}=\frac{d\vec{v}}{dt}$ 由相似形 $\frac{dv}{v}=\frac{v\,dt}{R}$

∴ $\vec{a_n}=\frac{v^2}{R}\hat{n}$.

若$|\vec{a_t}|$为一常量，则 $\omega=\beta t+\omega_0$，$\theta=\theta_0+\omega_0 t+\frac{1}{2}\beta t^2$.

4). $F=\frac{GMm}{r^2}$的椭圆.



设v_θ为横向速度，v_r为径向速度.

$$\begin{cases} L=mv_\theta r \\ E=\frac{1}{2}mv^2-\frac{GMm}{r}=\frac{1}{2}m(v_\theta^2+v_r^2)-\frac{GMm}{r} \end{cases}$$

在AB两处，$v_\theta=v$，$v=\frac{L}{mr_0}$.

∴ $E=\frac{1}{2}m\frac{L^2}{m^2r_0^2}-\frac{GMm}{r_0}$ 整理得 $r_0^2+\frac{GMm}{E}r_0-\frac{L^2}{2mE}=0$.

又∵在AB两 $r_0=a+c$或$a-c$

$$∴\begin{cases} r_{0_1}+r_{0_2}=2a=-\frac{GMm}{E} \\ r_{0_1}\cdot r_{0_2}=a^2-c^2=b^2=-\frac{L^2}{2mE}. \end{cases}$$

$$解得\begin{cases} E=-\frac{GMm}{2a} \\ L=mb\sqrt{\frac{GM}{a}} \end{cases} \Rightarrow \begin{cases} a=-\frac{GMm}{2E} \\ b=\sqrt{-\frac{L^2}{2mE}} \end{cases}$$

故椭圆方程为 $\frac{x^2}{\left(-\frac{GMm}{2E}\right)^2}+\frac{y^2}{\left(\sqrt{-\frac{L^2}{2mE}}\right)^2}=1$

曲率半径：$\frac{GMm}{r^2}\cos\theta = m\frac{v^2}{\rho}$

而 $mv\cos\theta\, r = mb\sqrt{\frac{GM}{a}}$，∴ $\cos\theta = \frac{b}{vr}\sqrt{\frac{GM}{a}}$

∴ $\rho = \frac{r^2v^2}{GM\cos\theta} = \frac{r^3v^3}{GMb\sqrt{\frac{GM}{a}}}$

而 $\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$，∴ $v = \sqrt{GM}\sqrt{\frac{2}{r}-\frac{1}{a}}$

∴ $\rho = \frac{r^3(GM)^{\frac{3}{2}}(\frac{2}{r}-\frac{1}{a})^{\frac{3}{2}}}{(GM)^{\frac{3}{2}}\, b\sqrt{\frac{1}{a}}} = \frac{r^3}{b}\sqrt{(\frac{2}{r}-\frac{1}{a})^3 \cdot a}$

A点：$\rho_A = \frac{(a-c)^3}{b}\sqrt{(\frac{2}{a-c}-\frac{1}{a})^3 \cdot a} = \frac{(a-c)^3}{ab}\sqrt{\frac{(a+c)^3}{(a-c)^3}} = \frac{b^2}{a}$

B点：$\rho_B = \frac{(a+c)^3}{b}\sqrt{(\frac{2}{a+c}-\frac{1}{a})^3 a} = \frac{(a+c)^3}{ab}\sqrt{\frac{(a-c)^3}{(a+c)^3}} = \frac{b^2}{a}$

C点：$\rho_C = \frac{a^3}{b}\sqrt{(\frac{2}{a}-\frac{1}{a})^3 a} = \frac{a^2}{b}$.

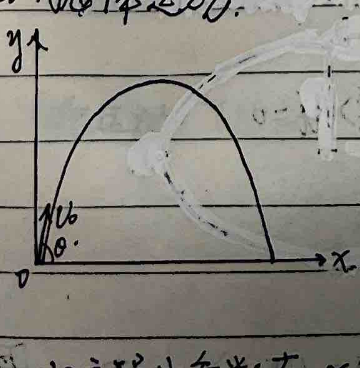
周期：由于单位时间矢径扫过的面积 $\frac{ds}{dt} = \frac{1}{2}|\vec{r}\times\vec{v}|\cdot dt/dt = \frac{1}{2}rv_\theta = \frac{1}{2}rv\cos\theta$.

∵ $\cos\theta = \frac{b}{vr}\sqrt{\frac{GM}{a}}$

∴ $\frac{ds}{dt} = \frac{1}{2}rv\frac{b}{vr}\sqrt{\frac{GM}{a}} = \frac{b}{2}\sqrt{\frac{GM}{a}}$.

∴ $\frac{\pi ab}{T} = \frac{b}{2}\sqrt{\frac{GM}{a}}$ ∴ $T = 2\pi\sqrt{\frac{a^3}{GM}}$.

5). 抛体运动：



$v_x = v_0\cos\theta$，$v_y = v_0\sin\theta - gt$

$x = v_0\cos\theta\, t$，$y = v_0\sin\theta\, t - \frac{1}{2}gt^2$.

射程 $x = \frac{v_0^2\sin 2\theta}{g}$，射高：$y = \frac{v_0^2\sin^2\theta}{2g}$.

轨道方程：$x = v_0\cos\theta\, t$ 和 $y = v_0\sin\theta\, t - \frac{1}{2}gt^2$ 消去t得

$y = -\frac{g}{2v_0^2\cos^2\theta}x^2 - \frac{\sin\theta}{\cos\theta}x = -\frac{g(1+tg^2\theta)}{2v_0^2}x^2 + tg\theta\, x$.

∴ 此方程中参数有 x、y、v_0、$tg\theta$（知三可求四，知二可求最）.

曲率半径：$\rho = \frac{v^2}{a_n} = \frac{v_0^2 - 2gH}{g\frac{v_x}{\sqrt{v_x^2+v_y^2}}} = \frac{(v_0^2 - 2v_0 g\sin\theta\, t + g^2t^2)^{\frac{3}{2}}}{g\cdot v_0\cos\theta}$.

包络线方程1：等初速度，各方向.

$$\text{最高点}\begin{cases} x = \dfrac{v_0^2\sin 2\theta}{2g} \\ y = \dfrac{v_0^2\sin^2\theta}{2g} = \dfrac{v_0^2}{4g}(1-\cos 2\theta) \end{cases}$$

消去θ，得 $\left(\dfrac{2g}{v_0^2}x\right)^2 + \left(1-\dfrac{4g}{v_0^2}y\right)^2 = 1$.

变形得 $\dfrac{x^2}{\left(\frac{v_0^2}{2g}\right)^2} + \dfrac{\left(y-v_0^2/(4g)\right)^2}{\left(\frac{v_0^2}{4g}\right)^2} = 1$

中心坐标 $(0, \frac{v_0^2}{4g})$，$a = \frac{v_0^2}{2g}$，$b = \frac{v_0^2}{4g}$.

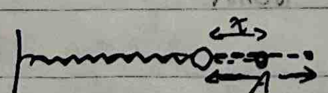
包络线方程2：同方向，各初速度.

$$\text{最高点}\begin{cases} x = \dfrac{v_0^2\sin 2\theta}{2g} \\ y = \dfrac{v_0^2}{4g}(1-\cos 2\theta) \end{cases}$$

消去v_0^2，$\dfrac{y}{x} = \dfrac{1}{2}\cdot\dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{1}{2}tg\theta$

(6) 简谐运动.

<1> 弹簧振子：



$\vec{F} = -k\vec{x} = m\dfrac{d^2\vec{x}}{dt^2}$

$\therefore kx + m\dfrac{d^2x}{dt^2} = 0$. $\therefore kx\,dx + m\,d\left(\dfrac{dx}{dt}\right)\dfrac{dx}{dt} = 0$

积分得 $\frac{1}{2}kx^2 - \frac{1}{2}kA^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = 0$.

$\therefore \dfrac{dx}{\sqrt{A^2-x^2}} = \sqrt{\dfrac{k}{m}}\,dt$.

令 $x = A\cos\theta$，代入上式得 $\theta = -\sqrt{\frac{k}{m}}\,t$ $\therefore x = A\cos(\sqrt{\frac{k}{m}}\,t)$ $\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$.

$$\begin{cases} v = \dfrac{dx}{dt} = -A\sqrt{\dfrac{k}{m}}\sin(\sqrt{\dfrac{k}{m}}\,t) \\ a = \dfrac{dv}{dt} = -A\dfrac{k}{m}\cos(\sqrt{\dfrac{k}{m}}\,t) \end{cases}$$

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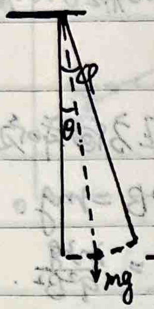
<2> 单摆



$$\begin{cases} \vec{F} = -mg\sin\theta\,\vec{i} \approx -mg\theta\,\vec{i} = m\dfrac{d\vec{v}}{dt} \\ mgl(1-\cos\varphi) = \dfrac{1}{2}mv^2 + mgl(1-\cos\theta) \end{cases}$$

$\therefore v = \sqrt{2gl(\cos\theta-\cos\varphi)} \approx \sqrt{gl(\varphi^2-\theta^2)}$

令 $\theta = \varphi\cos\alpha$，代入上式得 $v = \sqrt{gl}\,\varphi\sin\alpha$

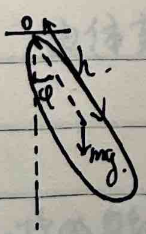
$\therefore dv = -\dfrac{\sqrt{gl}\,\theta\, d\theta}{\sqrt{\varphi^2-\theta^2}}$ 且 $dv = -g\theta\, dt$.

$\therefore dt = \sqrt{\dfrac{l}{g}}\dfrac{d\theta}{\sqrt{\varphi^2-\theta^2}} = \sqrt{\dfrac{l}{g}}\dfrac{d\theta}{\varphi\sin\alpha} = \sqrt{\dfrac{l}{g}}\cdot\dfrac{-\varphi\sin\alpha\, d\alpha}{\varphi\sin\alpha} = -\sqrt{\dfrac{l}{g}}\,d\alpha$

$\therefore \alpha = -\sqrt{\dfrac{g}{l}}\,t$. 代入前式得

$$\begin{cases} \theta = \varphi\cos\left(\sqrt{\dfrac{g}{l}}\,t\right) \Rightarrow T = 2\pi\sqrt{\dfrac{l}{g}} \\ v = -\varphi\sqrt{gl}\,\sin\left(\sqrt{\dfrac{g}{l}}\,t\right) \\ a = -\varphi g\cos\left(\sqrt{\dfrac{g}{l}}\,t\right). \end{cases}$$

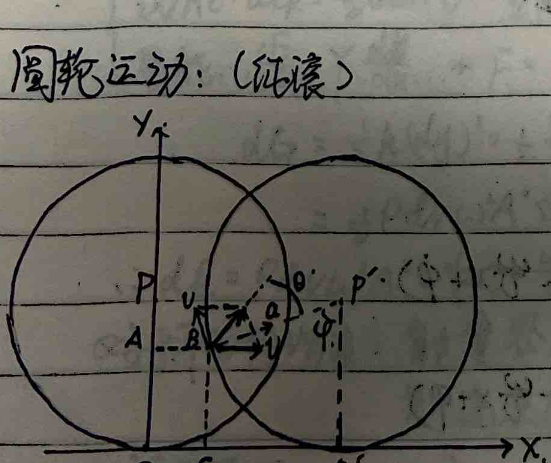
<3> 复摆



$$\begin{cases} M \approx -mgh\theta = I\dfrac{d\omega}{dt} \\ mgh(1-\cos\varphi) = \dfrac{1}{2}I\omega^2 + mgh(1-\cos\theta) \end{cases}$$

同理 $T = 2\pi\sqrt{\dfrac{I}{mgh}}$.

7). 圆轮运动:（纯滚）



在梯形 $ABP'P$ 中: $AB = x = PP' - R\sin\varphi = R(\varphi - \sin\varphi)$

在梯形 $BCO'P'$ 中: $BC = y = P'O' - R\cos\varphi = R(1-\cos\varphi)$.

曲率半径: $a = \dfrac{d|v\vec{\pi}_{切} + v\vec{\pi}_{摆}|}{dt}$

$= \dfrac{d|v\vec{\pi}_{切}|}{dt} = \dfrac{v^2}{R}$, 而 $v_{合}^2 = 2v^2(1+\cos\varphi)$.

$\therefore \rho = \dfrac{2v^2(1+\cos\varphi)}{\frac{v^2}{R}\sin\theta} = \dfrac{2v^2(1+\cos\varphi)}{\frac{v^2}{R}\sin\frac{\varphi}{2}} = -4R\dfrac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}$.

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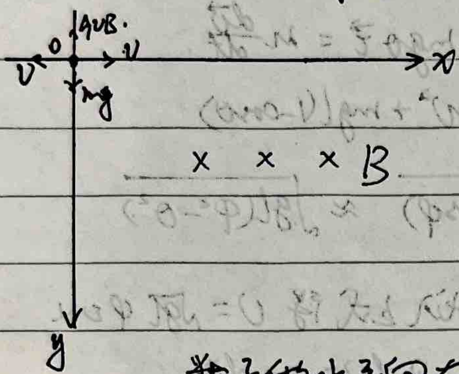
8). 重力场与磁场结合问题.



质量为m，带电量为+q的粒子从原点无初速度释放.

给粒子附加沿x轴正方向和负方向各一个速度v，使 $qvB = mg$.

$\therefore v = \dfrac{mg}{qB}$，$R = \dfrac{mv}{qB} = \dfrac{m^2 g}{q^2 B^2}$.

粒子做水平向右匀速直线运动和匀速圆周运动.

$\therefore x = vt - R\sin\omega t$，$y = R - R\cos\omega t$.

$\because v = \dfrac{mg}{qB} = \dfrac{qBR}{m}$ $\therefore \omega = \dfrac{v}{R} = \dfrac{qB}{m}$.

$$\therefore \begin{cases} x = \dfrac{mg}{qB}t - R\sin\left(\dfrac{qB}{m}t\right) = \dfrac{mg}{qB}t - \dfrac{m^2 g}{q^2B^2}\sin\left(\dfrac{qB}{m}t\right) \\ y = R - R\cos\left(\dfrac{qB}{m}t\right) = \dfrac{m^2 g}{q^2B^2} - \dfrac{m^2 g}{q^2B^2}\cos\left(\dfrac{qB}{m}t\right) \end{cases}$$

9) 波动.

横波：振动方向与传播方向垂直（固体中传播）

纵波：振动方向与传播方向相同（固、液、气中传播）.

$$y(x,t) = A\cos\left(\omega t - \frac{\omega}{v}x + \varphi\right)$$
$$= A\cos(\omega t - kx + \varphi) \qquad k\text{:波矢.}$$
$$= A\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda} + \frac{\varphi}{2\pi}\right).$$

当 $x = x_0$ 时，方程表示该点振动方程.

$$y(t) = A\cos\left(\omega t - \frac{\omega}{v}x_0 + \varphi\right)$$

当 $t = t_0$ 时，方程表示该时刻波形.

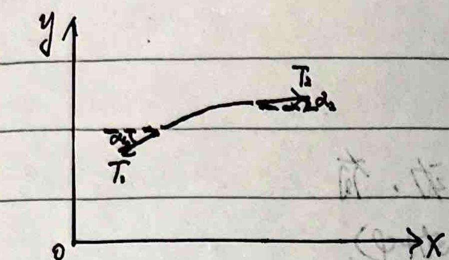
$$y(x) = A\cos\left(\omega t_0 - \frac{\omega}{v}x + \varphi\right)$$

<1> 波速：$\because \dfrac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos\left(\omega t - \dfrac{\omega}{v}x + \varphi\right)$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -\frac{\omega^2}{v^2}A\cos\left(\omega t - \frac{\omega}{v}x + \varphi\right)$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}.$$

轻质柔弦的横波方程：



$$\begin{cases} T_2\sin\alpha_2 - T_1\sin\alpha_1 = \rho\, ds\,\frac{\partial^2 y}{\partial t^2}. \\ T_2\cos\alpha_2 - T_1\cos\alpha_1 = 0. \end{cases}$$

$$\begin{cases} \cos\alpha_1 \approx \cos\alpha_2 \approx 1 \\ \sin\alpha_1 \approx tg\,\alpha_1 = \frac{\partial y}{\partial x}\Big|_x \quad \sin\alpha_2 \approx tg\,\alpha_2 = \frac{\partial y}{\partial x}\Big|_{x+dx} \end{cases}$$

$\therefore \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$

$\therefore v = \sqrt{\frac{T}{\rho}}$

稀薄气体中的声速：

气体可认为在传播过程中经历绝热过程. $PV^{\gamma} = C$

$\therefore V^{\gamma} dp + \gamma P V^{\gamma-1} dV = 0 \Rightarrow \frac{dp}{dV} = -\gamma\frac{p}{V}$.

由体变模量定义, $dp = -B\frac{dV}{V}$ $\therefore B = -V\frac{dp}{dV} = \gamma p$.

$\therefore v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{\mu}}$.

(2) 波的能量和能流.

$$\begin{cases} \text{动能}: dE_k = \frac{1}{2}dm\cdot v^2 = \frac{1}{2}\rho dV\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho dV \omega^2 A^2 \sin^2\left(\omega t - \frac{\omega}{u}x\right) \\ \text{势能}: \frac{F}{S} = Y\frac{dy}{dx} \Rightarrow F = \frac{YS}{dx}dy = k\,dy. \end{cases}$$

$dE_p = \frac{1}{2}k(dy)^2 = \frac{1}{2}\frac{YS}{dx}(dy)^2 = \frac{1}{2}\left(\frac{dy}{dx}\right)^2(S\,dx)\cdot Y = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 Y dV$

$= \frac{1}{2}\rho dV\omega^2 A^2\sin^2\left(\omega t - \frac{\omega}{u}x\right)$

$\therefore dE = \rho dV\omega^2 A^2 \sin^2\left(\omega t - \frac{\omega}{u}x\right)$.

对于平面简谐波：能量密度 $\varepsilon = \frac{dE}{dV} = \rho\omega^2 A^2 \sin^2\left(\omega t - \frac{\omega}{u}x\right)$.

($\therefore$) 能流密度 $I = \varepsilon v$ $\therefore \bar{I} = \bar{\varepsilon} v = \frac{1}{2}\rho A^2\omega^2 v$.

对于球面波：$I_1 4\pi r_1^2 = I_2 4\pi r_2^2$. $\therefore \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} = \frac{A_1^2}{A_2^2}$. $\therefore y = \frac{A_0 r_0}{r}\cos(\omega t - kr)$

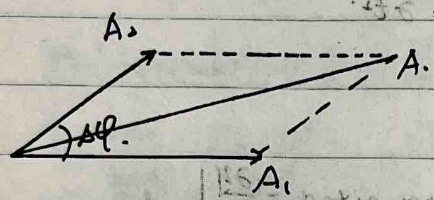
〈3〉波的干涉：

$$y_1(x,t)=A_1\cos(\omega t-\frac{\omega}{v}x_1+\varphi_1)$$

$$y_2(x,t)=A_2\cos(\omega t-\frac{\omega}{v}x_2+\varphi_2)$$

在某点p，两列波的合成仍为简谐运动，有

$$y(t)=y_1(x,t)+y_2(x,t)=A\cos(\omega t+\varphi)$$



$$\Delta\varphi=(\varphi_1-\varphi_2)-\frac{2\pi}{\lambda}(x_1-x_2)$$

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\varphi}$$

当$\Delta\varphi=2k\pi$时，干涉相长，$A=A_1+A_2$.

当$\Delta\varphi=(2k+1)\pi$时，干涉相消，$A=|A_1-A_2|$.

〈4〉驻波：实质为波的干涉

左行波 $y_1(x,t)=A\cos(\omega t+\frac{2\pi x}{\lambda})$

右行波 $y_2(x,t)=A\cos(\omega t-\frac{2\pi x}{\lambda})$

$\therefore y(x,t)=y_1(x,t)+y_2(x,t)=2A\cos\frac{2\pi x}{\lambda}\cos\omega t$.

当$\frac{2\pi x}{\lambda}=k\pi$，即$x=k\frac{\lambda}{2}$时，振幅取得最大，为波腹.

当$\frac{2\pi x}{\lambda}=(2k+1)\frac{\pi}{2}$，即$x=(2k+1)\frac{\lambda}{4}$时，振幅为零，为波节.

(※) 绳波中两个端点必然是驻波节点，故绳长$L=k\cdot\frac{\lambda}{2}$

核外电子绕原子核的运动为物质波，轨道周长满足 $2\pi R=k\lambda$.

④ 质点动力学.

瞬时效应：$\vec{F}=m\vec{a}=m\frac{d\vec{v}}{dt}$.

时间积累：$\vec{F}dt=md\vec{v}\Rightarrow\int_{t_1}^{t_2}\vec{F}dt=m(\vec{v}_2-\vec{v}_1)$.

空间积累：$\vec{F}d\vec{r}=m\vec{v}d\vec{v}\Rightarrow\int_{r_1}^{r_2}\vec{F}d\vec{r}=\frac{1}{2}m(v_2^2-v_1^2)$.

$P=\frac{dW}{dt}=\vec{F}\cdot\vec{v}$.

⑤刚体动力学： $\vec{M}=\vec{r}\times\vec{F}$

瞬时效应：$M=I\beta=I\frac{d\omega}{dt}$ （$I=\int r^2dm$，r：dm到转轴的距离）.

时间积累：$Mdt=Id\omega \Rightarrow \int_{t_1}^{t_2}Mdt=I(\omega_2-\omega_1)$

空间积累：$Md\theta=I\omega d\omega\Rightarrow\int_{\theta_1}^{\theta_2}Md\theta=\frac{1}{2}I(\omega_2^2-\omega_1^2)$

$p=\frac{dW}{dt}=\vec{M}\cdot\vec{\omega}$.

1) 转动惯量平行轴定理：$I=I_c+ml^2$

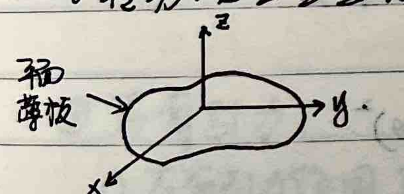
（I_c：刚体对过质心转轴的转动惯量，

I：刚体对与过质心转轴相距l的另一条平行转轴的转动惯量）

$I=\int r^2dm=\int(\vec{r_c}-\vec{l})^2dm=I_c+ml^2-2\vec{l}\cdot\int\vec{r_c}dm$

$=I_c+ml^2$.

2) 转动惯量垂直轴定理：$I_z=I_x+I_y$.

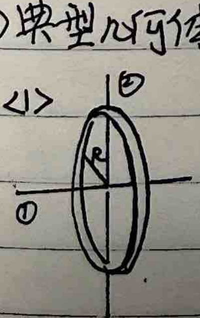


$I_z=\int r^2dm=\int(x\vec{i}+y\vec{j})^2dm$

$=\int(x^2+y^2)dm=I_x+I_y$.

3) 典型几何体转动惯量：

<1>



$I_1=\int r^2dm=mR^2$.

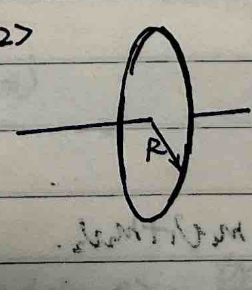
$I_2=4\int_0^{\frac{\pi}{2}}r^2\frac{Rd\theta}{2\pi R}\cdot m$

$=4\int_0^{\frac{\pi}{2}}\frac{R^2\cos^2\theta d\theta}{2\pi}\cdot m$

$=\frac{2mR^2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta}{2}\cdot d\theta$

$=\frac{1}{2}mR^2$.

<2>



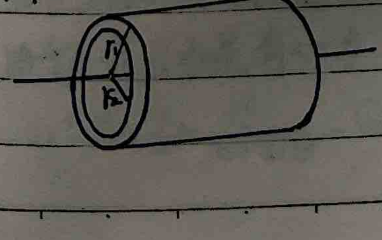
$I=\int r^2dm$

$=\int_0^R r^2\cdot\frac{2\pi rdr}{\pi R^2}\cdot m$

$=\frac{2m}{R^2}\int_0^R r^3dr$

$=\frac{1}{2}mR^2$.

<3>.

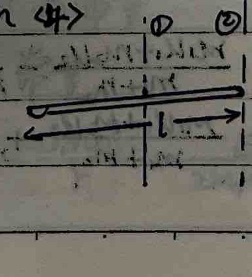


$I=\int_{r_2}^{r_1}\frac{2\pi rdr}{\pi(r_1^2-r_2^2)}\cdot m$

$=\frac{2m}{r_1^2-r_2^2}\int_{r_2}^{r_1}r^3dr$

$=\frac{1}{2}m(r_1^2+r_2^2)$

<4>



$I_1=\int_{-\frac{l}{2}}^{\frac{l}{2}}r^2\cdot\frac{dr}{l}m$

$=\frac{1}{12}ml^2$

$I_2=I_1+m(\frac{l}{2})^2$

$=\frac{1}{3}ml^2$.

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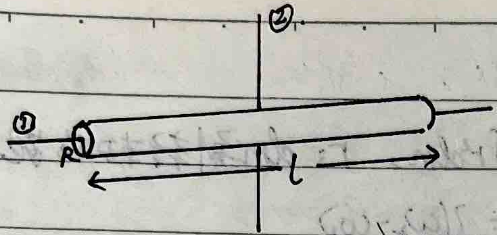
<5>.



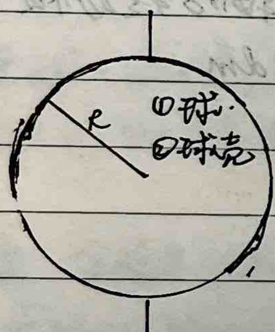
$$I_1 = \int_0^R r^2 \cdot \frac{2\pi r\,dr \cdot l}{\pi R^2 l} m = \frac{1}{2} mR^2$$

$$I_2 = \int \left[\frac{dm\, l^2}{12} + dm (R\sin\theta)^2\right] = \frac{1}{12} ml^2 + 2\int_0^{\frac{\pi}{2}} \frac{(2R\cos\theta)(R\,d\theta\cos\theta)}{\pi R^2} m (R\sin\theta)^2$$

$$= \frac{1}{12} ml^2 + \frac{4mR^2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta\, d\theta$$

$$= \frac{1}{12} ml^2 + \frac{1}{4} mR^2.$$

<6>.



$$I_1 = \int_0^{\frac{\pi}{2}} \frac{(2R\cos\theta)(2\pi R\sin\theta)(R\,d\theta\sin\theta)}{\frac{4}{3}\pi R^3} m (R\sin\theta)^2.$$

$$= 3mR^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \sin^3\theta\, d\theta = -3mR^2 \int_0^{\frac{\pi}{2}} \cos^2\theta (1-\cos^2\theta)\, d\cos\theta$$

$$= \frac{2}{5} mR^2.$$

$$I_2 = 2\int_0^{\frac{\pi}{2}} \frac{R\,d\theta \cdot (2\pi R\sin\theta)}{4\pi R^2} m (R\sin\theta)^2$$

$$= mR^2 \int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta = -mR^2 \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\, d\cos\theta$$

$$= \frac{2}{3} mR^2.$$

⑤ 碰撞.

1) 正碰:

$$\begin{cases} m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2. \\ v_2 - v_1 = e(u_1 - u_2) \end{cases}$$

$$\begin{cases} e=1 \text{ 完全弹性体} \\ e\in(0,1) \text{ 一般非弹性体} \\ e=0 \text{ 完全非弹性体.} \end{cases}$$

解得: $$v_1 = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} - \frac{m_2}{m_1 + m_2} e(u_1 - u_2)$$

$$v_2 = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2} e(u_1 - u_2)$$

第一项为质心速度：$\frac{m_1u_1+m_2u_2}{m_1+m_2}$。由于动量守恒，这一项不变。

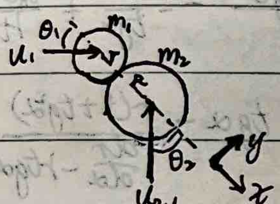
第二项为相对质心的运动速度，能量在质心系中由于二体作用而损耗。

损失的能量 $\Delta E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-\frac{1}{2}m_1u_1^2-\frac{1}{2}m_2u_2^2$

$$=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2-\frac{1}{2}\frac{m_1m_2}{m_1+m_2}[e(u_1-u_2)]^2$$

$$=\frac{1}{2}(1-e^2)\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2.$$

2) 斜碰.



$$v_{2x}-v_{1x}=e(u_1\cos\theta_1-u_2\cos\theta_2).$$

$$m_1u_1\cos\theta_1+m_2u_2\cos\theta_2=m_1v_{1x}+m_2v_{2x}\cdots\cdots$$

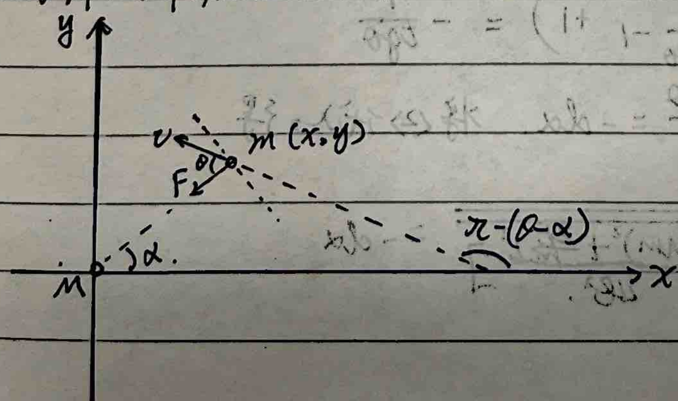
$$-m_1u_1\cos\theta_1+m_1v_{1x}=N\Delta t_1$$

$$-m_1u_1\sin\theta_1+m_1v_{1y}=\mu N\Delta t_2$$

$$m_1u_1\sin\theta_1+m_2u_2\sin\theta_2=m_1v_{1y}+m_2v_{2y}.$$

⑦ 万有引力.

1) 万有引力作用下质点的运动轨迹.



由上图，$\frac{y}{x}=tg\alpha$，$\frac{dy}{dx}=tg[\pi-(\theta-\alpha)]=tg(\alpha-\theta)=\frac{tg\alpha-tg\theta}{1+tg\alpha tg\theta}$ <1>

由于万有引力是保守力：$E=\frac{1}{2}mv^2-\frac{GMm}{r}$ 为常量.

由于万有引力是有心力：$L=mvr\sin\theta$ 为常量.

$$\therefore E=\frac{1}{2}m\frac{L^2}{m^2r^2\sin^2\theta}-\frac{GMm}{r}=\frac{L^2(1+tg^2\theta)}{2mr^2tg^2\theta}-\frac{GMm}{r}=\frac{L^2}{2mr^2tg^2\theta}+\frac{L^2}{2mr^2}-\frac{GMm}{r}$$

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解得 $tg\theta = \frac{L^2}{2mr^2} \Big/ \left(E + \frac{GMm}{r} - \frac{L^2}{2mr^2}\right)$

$$= \frac{L^2}{2mr^2E + 2GMm^2r - L^2}$$

$$\therefore tg\theta = \frac{L}{\sqrt{2mE\left[\left(r + \frac{GMm}{2E}\right)^2 - \frac{(GMm)^2 + \frac{2E}{m}L^2}{4E^2}\right]}} \quad \langle 2 \rangle$$

由于 $y = r\sin\alpha$，$x = r\cos\alpha$。$\langle 1 \rangle$式变为

$$\frac{\sin\alpha\, dr + r\cos\alpha\, d\alpha}{\cos\alpha\, dr - r\sin\alpha\, d\alpha} = \frac{tg\alpha - \frac{1}{tg\alpha}(tg\alpha\, tg\theta + 1) + \frac{1}{tg\alpha}}{1 + tg\alpha\, tg\theta} = -\frac{1}{tg\alpha} + \frac{\frac{1}{tg\alpha} + tg\alpha}{1 + tg\alpha\, tg\theta}$$

$$= \frac{tg\alpha \frac{dr}{d\alpha} + r}{\frac{dr}{d\alpha} - r\,tg\alpha} = \frac{tg\alpha\left(\frac{dr}{d\alpha} - r\,tg\alpha\right) + r(1 + tg^2\alpha)}{\frac{dr}{d\alpha} - r\,tg\alpha} = tg\alpha + \frac{r(1 + tg^2\alpha)}{\frac{dr}{d\alpha} - r\,tg\alpha}$$

$\therefore tg\alpha + \frac{1}{tg\alpha} + \frac{r(1+tg^2\alpha)}{\frac{dr}{d\alpha} - r\,tg\alpha} = \frac{tg\alpha + \frac{1}{tg\alpha}}{1 + tg\alpha\, tg\theta}$，两边消去 $1 + tg^2\alpha$，得

$$\frac{1}{tg\alpha} + \frac{r}{\frac{dr}{d\alpha} - r\,tg\alpha} = \frac{\frac{1}{tg\alpha}}{1 + tg\alpha\, tg\theta}$$

$$\therefore \frac{dr}{d\alpha} = r\,tg\alpha\left(\frac{1}{\frac{1}{1 + tg\alpha\, tg\theta} - 1} + 1\right) = -\frac{r}{tg\theta}$$

分离变量得 $\frac{dr\, tg\theta}{r} = -d\alpha$。将 $\langle 2 \rangle$ 代入，得

$$\frac{L\, dr}{r\sqrt{2mE\left[\left(r + \frac{GMm}{2E}\right)^2 - \frac{(GMm)^2 + \frac{2E}{m}L^2}{4E^2}\right]}} = -d\alpha \quad \langle 3 \rangle$$

令 $u = \frac{1}{r}$，得

$$\frac{L\left(-\frac{du}{u^2}\right)}{\frac{1}{u}\sqrt{2mE\left[\left(\frac{1}{u} + \frac{GMm}{2E}\right)^2 - \frac{(GMm)^2 + \frac{2E}{m}L^2}{4E^2}\right]}} = -d\alpha$$

化简得 $$\frac{du}{\sqrt{\frac{2mEL^2 + (GMm^2)^2}{L^4} - \left(u - \frac{GMm^2}{L^2}\right)^2}} = d\alpha \quad \langle 4 \rangle$$

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令 $\left(u-\frac{GMm^2}{L^2}\right)=\sqrt{\frac{2mEL^2+(GMm^2)^2}{L^4}}\cos\beta$. 得.

$d\beta=-d\alpha$.

∴ $u-\frac{GMm^2}{L^2}=\sqrt{\frac{2mEL^2+(GMm^2)^2}{L^4}}\cos\alpha$.

将 $u=\frac{1}{\sqrt{x^2+y^2}}$ 和 $\cos\alpha=\frac{x}{\sqrt{x^2+y^2}}$ 代入上式，得.

$$\frac{1}{\sqrt{x^2+y^2}}=\frac{GMm^2}{L^2}+\frac{\sqrt{2mEL^2+(GMm^2)^2}}{L^2}\frac{x}{\sqrt{x^2+y^2}} \qquad \langle 5\rangle$$

整理得 $$\frac{y^2}{-\frac{L^2}{2mE}}+\frac{\left(x-\frac{\sqrt{(GMm^2)^2+2mEL^2}}{2mE}\right)^2}{\left(\frac{GMm}{2E}\right)^2}=1 \qquad \langle 6\rangle$$

1. 若 $E=0$，对于 $\langle 5\rangle$ 式有

$$\frac{1}{\sqrt{x^2+y^2}}=\frac{GMm^2}{L^2}+\frac{GMm^2}{L^2}\frac{x}{\sqrt{x^2+y^2}}$$，化简得

$$1-\frac{2GMm^2}{L^2}x=\left(\frac{GMm^2}{L^2}\right)^2y^2 \qquad \langle 7\rangle$$

此为一抛物线方程.

2. 若 $E>0$，对于 $\langle 6\rangle$ 式有. $-\frac{L^2}{2mE}<0$

∴ $$\frac{\left(x-\frac{\sqrt{(GMm^2)^2+2mEL^2}}{2mE}\right)^2}{\left(\frac{GMm}{2E}\right)^2}-\frac{y^2}{\left(\sqrt{\frac{L^2}{2mE}}\right)^2}=1.$$ 双曲线 $\langle 8\rangle$.

3. 若 $E<0$，对于 $\langle 6\rangle$ 式有，$-\frac{L^2}{2mE}>0$.

∴ $$\frac{\left(x-\frac{\sqrt{(GMm^2)^2+2mEL^2}}{2mE}\right)^2}{\left(-\frac{GMm}{2E}\right)^2}+\frac{y^2}{\left(\sqrt{-\frac{L^2}{2mE}}\right)^2}=1.$$ 椭圆 $\langle 9\rangle$.

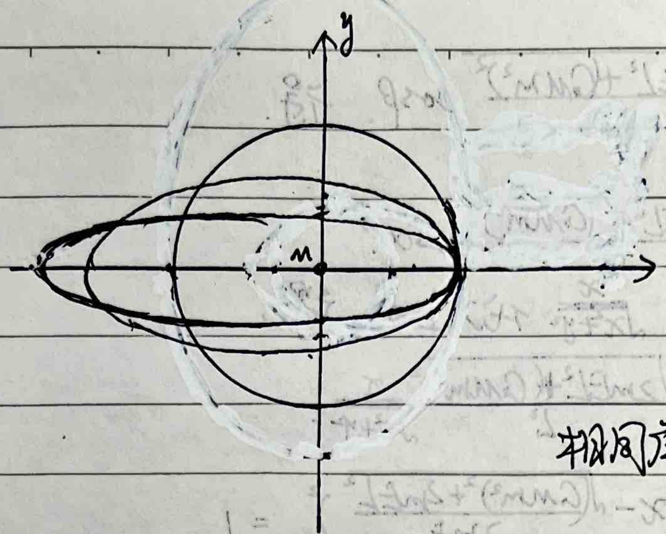
△ 当 $(GMm^2)^2+2mEL^2=0$，即 $E=-\frac{(GMm^2)^2}{2mL^2}=-\frac{(GMm)^2}{\frac{2L^2}{m}}$ 时.

∴ $\langle 9\rangle$ 式变为 $x^2+y^2=\left(-\frac{GMm}{2E}\right)^2$，

此为一圆方程

△ 当 $E<-\frac{GMm}{\frac{2L^2}{m}}$ 时，此情况不存在，$\Delta<0$.

综上，所有情况如下页图.



相同角动量不同能量对应的轨道

2）引力势能（无限远处 $E_p=0$）

① 均匀球体（半径 R，质量 M）

1）当 $r\geq R$ 时

$$E_p=\int_\infty^r \frac{GMm}{r^2}dr=-\frac{GMm}{r}\Big|_\infty^r=-\frac{GMm}{r}.$$

2）当 $r<R$ 时

$$E_p=\int_\infty^R \frac{GMm}{r^2}dr+\int_R^r \frac{G(\frac{r^3}{R^3}M)m}{r^2}dr=-\frac{GMm}{R}-\frac{GMm(R^2-r^2)}{2R^3}.$$

② 薄球壳（半径 R，质量 M）

1）当 $r\geq R$ 时

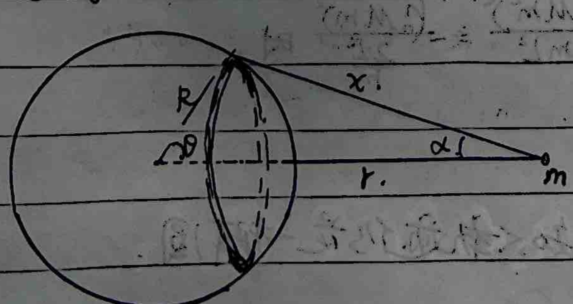
$$E_p=-\frac{GMm}{r}$$

2）当 $r<R$ 时

$$E_p=-\frac{GMm}{R}.$$

3）引力场强：高斯定理：$\frac{1}{4\pi G}\oint_S \vec{E}\cdot d\vec{S}=\sum_i m_i$.

① 薄球壳（半径 R 质量 M）



$$dE=G\cdot\frac{2\pi R\sin\theta\, Rd\theta\cdot\sigma}{x^2}\cos\alpha.$$

$$\because \cos\theta=\frac{x^2+r^2-R^2}{2rx},\quad x^2=R^2+r^2-2Rr\cos\theta$$

$$\therefore x\,dx=Rr\sin\theta\, d\theta.$$

$$\therefore dE=\frac{2\pi R^2G\sigma}{Rr}x\,dx\cdot\frac{x^2+r^2-R^2}{2rx}\cdot\frac{1}{x^2}$$

$$=\frac{2\pi R^2G\sigma}{2r^2R}\left(\frac{r^2-R^2}{x^2}+1\right)dx.$$

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1) 当$r \geq R$时

$\because \int_{r-R}^{r+R}(\frac{r^2-R^2}{x^2}+1)dx=4R \quad \therefore E=\frac{GM}{r^2}$.

2) 当$r<R$时

$\because \int_{R-r}^{R+r}(\frac{r^2-R^2}{x^2}+1)dx=0$， $\therefore E=0$.

② 均匀球体（半径R，质量M）

1) 当$r \geq R$时

$$E=\int_0^R G\frac{4\pi h^2 dh \rho}{r^2} = G\frac{\frac{4}{3}\pi R^3 \cdot \frac{M}{\frac{4}{3}\pi R^3}}{r^2} = \frac{GM}{r^2}.$$

2) 当$r<R$时

$$E=\frac{G\frac{r^3}{R^3}M}{r^2}=\frac{GMr}{R^3}.$$

4) 宇宙速度

① 第一宇宙速度：（环绕的最小速度）

$E=\frac{1}{2}mv^2-\frac{GMm}{r}$，而$\frac{mv^2}{r}=\frac{GMm}{r^2}$

$\therefore E=\frac{1}{2}mv^2-mv^2=-\frac{1}{2}mv^2$，

可见v越大E越小，对于半径为R的星球，$v_{max}=\sqrt{\frac{GM}{R}}$

$\therefore E_{min}=-\frac{GMm}{2R}$，此时绕体作圆周运动。

又$\because$地球上$g=\frac{GM}{R^2}$，$\therefore v_I=\sqrt{gR}=7.9km/s$

② 第二宇宙速度：（逃逸的最小速度）

$E=\frac{1}{2}mv^2-\frac{GMm}{r}$，当$E \geq 0$时，轨迹为抛物线或双曲线，即脱离环绕。

故$E=0$时，逃逸所需速率最小，对于地球$v_{II}=\sqrt{\frac{2GM}{R}}=11.2km/s$；

对于黑洞$v_{II}=\sqrt{\frac{2GM}{R_s}}=c$，故$R_s=\frac{2GM}{c^2}$（史瓦西半径）。

③ 第三宇宙速度：（飞出太阳系）

若地球不公转，则$\frac{1}{2}mv^2-\frac{GMm}{R_{日地}}=0$.

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得 $v=\sqrt{\frac{2GM}{R_{地}}}=42.2km/s$.

以地球为参考系（地球公转速 29.8 km/s），有

$$\frac{1}{2}mv_{Ⅲ}^2-\frac{GM_{地}m}{R_{地}}=\frac{1}{2}m(42.2km/s-29.8km/s)^2$$

解得 $v_{Ⅲ}=\sqrt{(42.2-29.8)^2+11.2^2}\,km/s=16.7km/s$.

专题：质心系

1. 定义：$\frac{\sum m_i\vec{r_{ci}}}{\sum m_i}=0$

求导得 $\frac{\sum m_i\vec{v_{ci}}}{\sum m_i}=0$，即 $\sum m_i\vec{v_{ci}}=0$，故质心系又称零动量系.

2. 柯尼希定理

孤立二体系统中质点1.2地面系中速度分别为 $\vec{v_1}$、$\vec{v_2}$。

在质心系中速度分别为 $\vec{v_1'}$、$\vec{v_2'}$，质心系相对地面速度为 $\vec{v_c}$.

∴ $\vec{v_1}=\vec{v_1'}+\vec{v_c}$，$\vec{v_2}=\vec{v_2'}+\vec{v_c}$

∴ 地面系中的动能 $E_k=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$

$=\frac{1}{2}m_1(\vec{v_1'}+\vec{v_c})^2+\frac{1}{2}m_2(\vec{v_2'}+\vec{v_c})^2$

$=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_1v_c^2+m_1\vec{v_1'}\cdot\vec{v_c}+\frac{1}{2}m_2v_2'^2+\frac{1}{2}m_2v_c^2+m_2\vec{v_2'}\cdot\vec{v_c}$

$=E_1+E_2+\frac{1}{2}(m_1+m_2)v_c^2+(m_1\vec{v_1'}+m_2\vec{v_2'})\cdot\vec{v_c}$.

∵ 质心系中总动量为零，∴ $m_1\vec{v_1'}+m_2\vec{v_2'}=0$，

∴ $E_k=E_1+E_2+\frac{1}{2}(m_1+m_2)v_c^2$.

其中：E_1、E_2 分别为1.2在质心系中的动能，$\vec{v_c}=\frac{m_1\vec{v_1}+m_2\vec{v_2}}{m_1+m_2}$.

推广到n质点孤立系统，$E_k=\frac{1}{2}m_cv_c^2+\sum E_{i质心系}$

设 $\vec{v_r}=\vec{v_1}-\vec{v_2}=\vec{v_1'}-\vec{v_2'}$ 为两质点的相对速度.

联立 $\vec{v_r}=\vec{v_1'}-\vec{v_2'}$，$m_1\vec{v_1'}+m_2\vec{v_2'}=0$，解得：

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$$\begin{cases} \vec{v_1}' = \dfrac{m_2\vec{u}}{m_1+m_2} \\ \vec{v_2}' = -\dfrac{m_1\vec{u}}{m_1+m_2} \end{cases}$$

$$\therefore E_k = \frac{1}{2}m_1\frac{m_2^2u^2}{(m_1+m_2)^2} + \frac{1}{2}m_2\frac{m_1^2u^2}{(m_1+m_2)^2} + \frac{1}{2}(m_1+m_2)v_c^2$$

$$= \frac{1}{2}\frac{m_1m_2}{m_1+m_2}u^2 + \frac{1}{2}(m_1+m_2)v_c^2.$$

注：① 对于动量守恒的孤立二体系统，相互作用只可能损失质心系中的动能，即 $\frac{1}{2}\frac{m_1m_2}{m_1+m_2}u^2$ 一项，而质心动能 $\frac{1}{2}(m_1+m_2)\left(\frac{m_1\vec{v_1}+m_2\vec{v_2}}{m_1+m_2}\right)^2$ 不变。

② 对于受到外力的二体系统，设 m_1 所受外力 $\vec{F_1}$，m_2 所受外力 $\vec{F_2}$。

则 $\vec{F_1}+\vec{F_2}=(m_1+m_2)\vec{a_c}=(m_1+m_2)\cdot\frac{d\vec{v_c}}{dt}$。这是对柯尼希定理中第二项的作用；而对第一项的作用，可由 $m_1\frac{d\vec{v_1}'}{dt}+m_2\frac{d\vec{v_2}'}{dt}=0$，

$\vec{F_1}+\vec{F_内}=m_1\frac{d\vec{v_1}}{dt}$，$\vec{F_2}-\vec{F_内}=m_2\frac{d\vec{v_2}}{dt}$，$\vec{v_1}'-\vec{v_2}'=\vec{v_1}-\vec{v_2}=\vec{u}$ 求得，

$$\therefore \frac{\vec{F_1}+\vec{F_内}}{m_1}-\frac{\vec{F_2}-\vec{F_内}}{m_2}=\frac{d(\vec{v_1}-\vec{v_2})}{dt}=\frac{d\vec{u}}{dt}.$$

$$=\frac{\vec{F_1}}{m_1}-\frac{\vec{F_2}}{m_2}+\frac{m_1+m_2}{m_1m_2}\cdot\vec{F_内}.$$ 可以看出，若 $\vec{F_1}$ 和 $\vec{F_2}$ 为0，有 $\vec{F_内}=\frac{m_1m_2}{m_1+m_2}\frac{d\vec{u}}{dt}$。

③ 刚体的转动不存在柯尼希定理。

地面系中 $E_k=\frac{1}{2}I_1\omega_1^2+\frac{1}{2}I_2\omega_2^2$

质心系中 $E_k'=\frac{1}{2}I_1'\omega_1^2+\frac{1}{2}I_2'\omega_2^2$，$I_1'\vec{\omega_1}+I_2'\vec{\omega_2}\neq0$

3. 质心系角动量定理：

$$\frac{d\vec{L}}{dt}=\vec{M}+\sum\vec{r_{ci}}\times\vec{f_{ci}}(\text{惯性力})=\vec{M}+\sum\vec{r_{ci}}\times(-m_i\vec{a_c})$$

$$=\vec{M}-\left(\sum m_i\vec{r_{ci}}\right)\times\vec{a_c}=\vec{M}.$$

4. 角动量柯尼希定理：

设 $\vec{r_i}$ 和 $\vec{v_i}$ 分别是刚体上 m_i 对 P 点的位矢和速度，

$$\vec{L}=\sum(\vec{r_i}\times m_i\vec{v_i})$$

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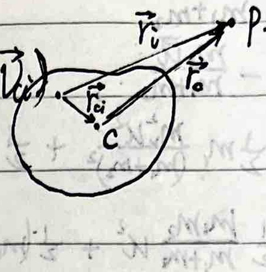
$= \sum[(\vec{r}_c + \vec{r}_{ci}) \times m_i(\vec{v}_c + \vec{v}_{ci})]$.



$= \sum(\vec{r}_c \times m_i\vec{v}_c + \vec{r}_c \times m_i\vec{v}_{ci} + \vec{r}_{ci} \times m_i\vec{v}_c + \vec{r}_{ci} \times m_i\vec{v}_{ci})$

$= \vec{r}_c \times m\vec{v}_c + \vec{r}_c \times (\sum m_i\vec{v}_{ci})$

$+ (\sum m_i\vec{r}_{ci}) \times \vec{v}_c + \sum(\vec{r}_{ci} \times m_i\vec{v}_{ci})$.

$\because \sum m_i\vec{v}_{ci} = 0,\ \sum m_i\vec{r}_{ci} = 0$

$\therefore \vec{L} = \vec{r}_c \times m\vec{v}_c + \sum(\vec{r}_{ci} \times m_i\vec{v}_{ci})$

质心角动量　　质心系中的角动量.

5. 质心系中惯性力做功为零，惯性力力矩做功为零.

$W_{惯} = \int(-m_i\vec{a}_c)\cdot d\vec{r}_{ci}$　　$W_{惯} = \int|\vec{r}_{ci} \times (-m_i\vec{a}_c)|\cdot d\vec{\theta}$

$= -\vec{a}_c\int m_i d\vec{r}_{ci} = 0$.　　$= -\int|(m_i\vec{r}_{ci}) \times \vec{a}_c|\, d\vec{\theta} = 0$.

6. 应用典例.

① 碰撞.

质心系中，$m_1v_{1c} + m_2v_{2c} = m_1u_{1c} + m_2u_{2c} = 0$.

$v_1 - v_2 = v_{1c} - v_{2c} = e(u_{2c} - u_{1c}) = e(u_2 - u_1)$

解得 $\begin{cases} v_{1c} = -eu_{1c} \\ v_{2c} = -eu_{2c} \end{cases}$

② 天体运动周期.

$G\frac{Mm}{R^2} = \frac{Mm}{M+m}\frac{4\pi^2}{T^2}R$

$\therefore T = 2\pi\sqrt{\frac{R^3}{G(M+m)}}$

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二. 狭义相对论

1. 伽利略变换：

① $$\begin{cases} x = x' + vt' \\ y = y' \\ z = z' \\ t = t' \end{cases}$$

② $\vec{r} = \vec{r'} + \vec{r}_{相}$　绝对 = 相对 + 牵连.

$\therefore \vec{v} = \vec{v'} + \vec{v}_{相}$

$\therefore \vec{a} = \vec{a'} + 0$　$\because m = m'$　$\therefore \vec{F} = \vec{F'}$

平移惯性力：$\vec{F} = -m\vec{a}$

转动惯性力：$\vec{F} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

科里奥利力：$\vec{F} = 2m\vec{\omega}\times\vec{v}_{相}$.

③ 多普勒效应：

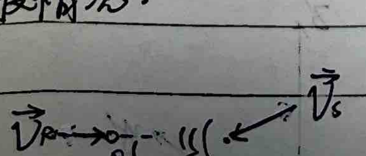
1) 观察者以 v_R 运动.

$$\nu_R = \frac{v+v_R}{\lambda} = \frac{v+v_R}{v/\nu} = \frac{v+v_R}{v}\cdot\nu$$ （v_R 相向为正远离为负）

2) 波源相对介质以 v_s 运动.

$$\nu_s = \frac{v}{\lambda-\frac{v_s}{\nu}} = \frac{v}{\frac{v-v_s}{\nu}} = \frac{v}{v-v_s}\cdot\nu$$ （v_s 相向为正远离为负）

3) 一般情况.



$$\nu' = \frac{v+v_R\cos\theta}{v-v_s}\cdot\nu$$

只有纵向多普勒效应

没有横向多普勒效应.

2. 洛伦兹变换

① 基本假设：

1) 所有惯性系平权（惯性定律成立的参考系是惯性系）—— 相对性原理.

2) 所有惯性系中真空光速不变 —— 光速不变原理

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② 洛伦兹变换：保证所有物理定律具有协变性的坐标变换公式.

1) 设惯性系 K' 相对于惯性系 K 沿 x 方向以 v 运动.

$x=x'=0$ 时，$y=y'=0$，$z=z'=0$，$t=t'=0$.

2) 相对性原理要求，由 $K(x,y,z,t)$ 到 $K'(x',y',z',t')$ 的坐标变换应是对称的，即正、逆变换都是同一种函数. 此要求来源于空间的均匀性，满足这一要求的变换应是一次函数.

$$\begin{cases} x'=ax+bt \\ t'=px+qt \end{cases} \Rightarrow \begin{cases} x=\dfrac{q}{aq-bp}x'-\dfrac{b}{aq-bp}t' \\ t=\dfrac{p}{bp-aq}x'-\dfrac{a}{bp-aq}t' \end{cases}$$

由于相对运动只发生在 x 方向，故

$$\begin{cases} y'=y \\ z'=z \end{cases}$$

3) 光速不变原理要求，若在 $t=t'=0$ 时刻由原点发出一闪光，在两惯性系中有

$$\begin{cases} x^2+y^2+z^2-c^2t^2=0 \\ x'^2+y'^2+z'^2-c^2t'^2=0 \end{cases}$$

对于任意一事件，令 $x^2+y^2+z^2-c^2t^2=\lambda(x'^2+y'^2+z'^2-c^2t'^2+A)$

由对称性和连续性，得 $x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2$.

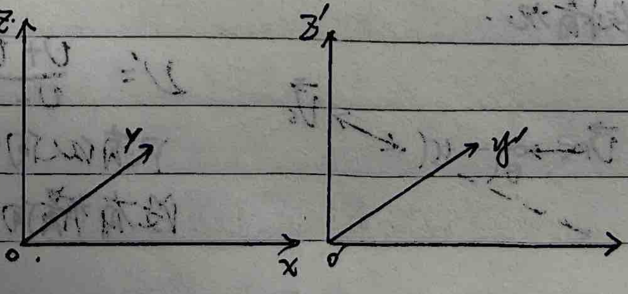
4) 在 K 系中，K 系的原点 O 满足

$$\begin{cases} \dfrac{dx'}{dt}=a\dfrac{dx}{dt}+b \\ \dfrac{dt'}{dt}=p\dfrac{dx}{dt}+q \end{cases}$$

$\because \dfrac{dx}{dt}\equiv 0$

$\therefore dx'=b\,dt,\ dt'=q\,dt$

$\therefore \dfrac{dx'}{dt'}=\dfrac{b}{q}=-v$



（K' 系中 K 系原点朝 $-x'$ 方向运动，故 v 为负）.

在 K' 系中，K' 系的原点 O' 满足

$$\begin{cases} \dfrac{dx}{dt'}=\dfrac{q}{aq-bp}\dfrac{dx'}{dt'}-\dfrac{b}{aq-bp} \\ \dfrac{dt}{dt'}=\dfrac{p}{bp-aq}\dfrac{dx'}{dt'}-\dfrac{a}{bp-aq} \end{cases}$$

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$\because \frac{dx'}{dt} \equiv 0$

$\therefore dx = -\frac{b}{aq-bp}dt'$, $dt = -\frac{a}{bp-aq}dt'$

$\therefore \frac{dx}{dt} = -\frac{b}{p} = v$（K系中K'系原点朝$+x$方向运动，故$v$为正）.

所以 $b=-av$ ，$a=q$. 代入变换式，得

$$\begin{cases} x' = ax - avt \\ t' = px + at \end{cases}$$

代入事件间隔恒等式，得

$$x^2+y^2+z^2-c^2t^2 = (ax-avt)^2+y^2+z^2-c^2(px+at)^2$$

化简得：

$$x^2-c^2t^2 = (a^2-c^2p^2)x^2+(a^2v^2-a^2c^2)t^2-(2a^2v+2apc^2)xt.$$

由于同类项前系数相等：

$$\begin{cases} 1 = a^2-c^2p^2 \\ -c^2 = a^2v^2-a^2c^2 \\ 0 = 2a^2v+2apc^2 \end{cases} \Longrightarrow \begin{cases} a = \frac{1}{\sqrt{1-v^2/c^2}} \\ p = -\frac{v}{c}\sqrt{\frac{1}{c^2-v^2}} \end{cases}$$ 代入变换式，得

$$\begin{cases} x' = \frac{1}{\sqrt{1-v^2/c^2}}(x-vt) \\ y' = y \\ z' = z \\ t' = \frac{1}{\sqrt{1-v^2/c^2}}(t-\frac{v}{c^2}x) \end{cases} \quad \begin{cases} x = \frac{1}{\sqrt{1-v^2/c^2}}(x'+vt') \\ y = y' \\ z = z' \\ t = \frac{1}{\sqrt{1-v^2/c^2}}(t'+\frac{v}{c^2}x') \end{cases}$$

注：(1) 注意推导的前提是 $x=x'=0$，$y=y'=0$，$z=z'=0$，$t=t'=0$

此前提不失一般性，但由原理推出的式子只适合这一前提，

对于 $t=t'=0$ 时刻原点不重合的情况而式不符合.

(2) 由于空间的均匀性，所有惯性系平权，只有线性的变换方程才能

体现这一原理.

(3) 光速不变原理本质要求是 $(dx)^2+(dy)^2+(dz)^2-(cdt)^2=(dx')^2+(dy')^2+(dz')^2-(cdt')^2$

$=ds$，$=\sqrt{c^2(d\tau)^2-(dl_0)^2}$ τ为固有时，l_0为固有长. $\because dl_0=0$ $\therefore d\tau = \frac{ds}{c}$

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③相对论时空观.

1)时序的相对性与因果关系.

设事件A、B在K'系中发生的地点与时间分别是(x_1', t_1')和(x_2', t_2').

则K系中，事件A、B发生的时间：

$$t_1 = \frac{t_1' + \frac{v}{c^2}x_1'}{\sqrt{1-v^2/c^2}}, \quad t_2 = \frac{t_2' + \frac{v}{c^2}x_2'}{\sqrt{1-v^2/c^2}}.$$

$$\therefore t_2 - t_1 = \frac{(t_2'-t_1') + \frac{v}{c^2}(x_2'-x_1')}{\sqrt{1-v^2/c^2}} = \frac{(t_2'-t_1')\left(1 + \frac{v}{c^2}\cdot\frac{x_2'-x_1'}{t_2'-t_1'}\right)}{\sqrt{1-v^2/c^2}}$$

类时间隔事件：$\frac{x_2'-x_1'}{t_2'-t_1'} > -c$ 可能有因果关系，有则因果不会倒置.
可能无因果关系，时间顺序不会颠倒.

类空间隔事件：$\frac{x_2'-x_1'}{t_2'-t_1'} < -c$ 无因果关系，时间顺序可能颠倒
(若有，则要么观察超光速要么因果佯逆超越)

2)①同时的相对性：

若K'系中在t'时刻x_1'和x_2'处同时发生了两事件A、B，在K系中，有

$$t_1 = \frac{t' + \frac{v}{c^2}x_1'}{\sqrt{1-v^2/c^2}}, \quad t_2 = \frac{t' + \frac{v}{c^2}x_2'}{\sqrt{1-v^2/c^2}}$$

对K系而言，这两事件不一定同时发生，A、B在K系中时间间隔为

$$\Delta t = t_2 - t_1 = \frac{v}{c^2}\frac{x_2'-x_1'}{\sqrt{1-v^2/c^2}}.$$

故只能在一个坐标系中对钟，或在两个坐标系中两个相接触点对钟.

②时间间隔相对性：

若K'系中在x'处于t_1'和t_2'时刻发生了两事件A、B，在K系中，有

$$t_1 = \frac{t_1' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}, \quad t_2 = \frac{t_2' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}$$

$$\Delta t = t_2 - t_1 = \frac{t_2'-t_1'}{\sqrt{1-v^2/c^2}} = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} > \Delta t'$$

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故在不同参考系中A、B时间间隔为固有时(原时间隔)最短.

③ 同地的相对性

若K系中在 x' 处于 t_1' 和 t_2' 时刻发生了两事件A、B，在K系中，有

$$x_1 = \frac{x'+vt_1'}{\sqrt{1-v^2/c^2}}, \quad x_2 = \frac{x'+vt_2'}{\sqrt{1-v^2/c^2}}$$

$$\Delta x = x_2 - x_1 = \frac{v(t_2'-t_1')}{\sqrt{1-v^2/c^2}}$$

对K系而言，这两事件不同地发生，原因是K相对K'有速度！

④ 长度的相对性.

在K'系中沿 x' 方向放有一静止直尺，两端坐标为 x_1'、x_2'，其固有长度 $l' = |x_1'-x_2'|$. 在K系中测量的时刻为 t，(在动参考系中异地对钟)，有

$$x_1' = \frac{x_1-vt}{\sqrt{1-v^2/c^2}}, \quad x_2' = \frac{x_2-vt}{\sqrt{1-v^2/c^2}}$$

故在K系中，$l = |x_1-x_2| = |x_1'-x_2'|\sqrt{1-v^2/c^2} = l'\sqrt{1-v^2/c^2}$.

但在静系K'中，x_1' 与 x_2' 两处时刻并不相同，有时差 $\Delta t' = \frac{1}{\sqrt{1-v^2/c^2}}(\Delta t - \frac{v}{c^2}\Delta x)$

由于在动系K中测量时 x_1，x_2 处时刻相同，故 $\Delta t = 0$ ∴ $\Delta t' = \frac{v}{c}\frac{l}{\sqrt{1-v^2/c^2}} = \frac{v}{c^2}l'$.

④ 四维矢量：[$u = (u_x, u_y, u_z, u_t)$：四维速度；$v = (v_x, v_y, v_z)$ 三维速度；V：参考系速]

1) 闵可夫斯基空间

$$\begin{cases} x' = \gamma[x + i\beta(ict)] \\ y' = y \\ z' = z \\ ict' = \gamma[(ict) - i\beta x] \end{cases}$$

洛仑兹不变量：

$$x'^2 + y'^2 + z'^2 + (ict')^2$$
$$= \gamma^2(x-\beta ct)^2 + y^2 + z^2 + \gamma^2(-1)(ct-\beta x)^2$$
$$= x^2 + y^2 + z^2 + (ict)^2 = -s^2.$$

2) 四维速度.

固有时 $d\tau = \frac{ds}{c}$

定义四维速度 $u_x = \frac{dx}{d\tau}$，$u_y = \frac{dy}{d\tau}$；$u_z = \frac{dz}{d\tau}$，$u_t = \frac{d(ict)}{d\tau}$.

∵ $u_x = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma\frac{dx}{dt} = \frac{v_x}{\sqrt{1-v^2/c^2}}$，同理，$u_y = \frac{v_y}{\sqrt{1-v^2/c^2}}$，$u_z = \frac{v_z}{\sqrt{1-v^2/c^2}}$

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以上 v_x、v_y、v_z 为 K系中物体的三维速度.

对于四维速度的最后一分量，$u_t = \frac{d(ict)}{dt}\frac{dt}{d\tau} = ic\gamma = \frac{ic}{\sqrt{1-v^2/c^2}}$.

$\therefore u = (u_x, u_y, u_z, u_t) = (\gamma v_x, \gamma v_y, \gamma v_z, ic\gamma)$.

由于 $dx, dy, dz, d(ict)$ 服从洛伦兹变换，且 $d\tau$ 是不变量，所以

$$\begin{cases} u_x' = \gamma_0(u_x + i\beta u_t) \\ u_y' = u_y \\ u_z' = u_z \\ u_t' = \gamma_0(u_t - i\beta u_x) \end{cases}$$

洛伦兹不变量：

$$u_x^2 + u_y^2 + u_z^2 + u_t^2 = \frac{v_x^2 + v_y^2 + v_z^2 - c^2}{1 - v^2/c^2} = \frac{v^2 - c^2}{1 - v^2/c^2} = -c^2.$$

$$\therefore \gamma' v_x' = \gamma_0 \gamma (v_x - c\beta)$$
$$\gamma' v_y' = \gamma v_y$$
$$\gamma' v_z' = \gamma v_z$$
$$c\gamma' = \gamma_0 \gamma (c - \beta v_x)$$

其中 $\gamma_0 = \frac{1}{\sqrt{1-\beta^2}}$，$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$（$v$ 为K系中速度）$\gamma' = \frac{1}{\sqrt{1-(v'/c)^2}}$（$v'$ 为K' 系中速度

$$\frac{\gamma_0\gamma}{\gamma'} = \frac{1}{1-\frac{\beta v_x}{c}} = \frac{1}{1-\frac{V v_x}{c^2}}$$

（$\beta = \frac{V}{c}$，V 为K与K' 的相对速度）.

将此式代入前三式，得.

$$\begin{cases} v_x' = \frac{\gamma_0\gamma}{\gamma'}(v_x - c\beta) = \frac{v_x - V}{1 - \frac{v_x V}{c^2}} \\ v_y' = \frac{\gamma}{\gamma'} v_y = \frac{v_y\sqrt{1-V^2/c^2}}{1-\frac{v_x V}{c^2}} \\ v_z' = \frac{\gamma}{\gamma'} v_z = \frac{v_z\sqrt{1-V^2/c^2}}{1-\frac{v_x V}{c^2}} \end{cases}$$

（速度变换律可直接将洛伦兹变换微分求得出，即 $dx = \frac{dx' + V dt'}{\sqrt{1-V^2/c^2}}$，$dy = dy'$，$dz = dz'$，$dt = \frac{dt' + \frac{V}{c^2}dx'}{\sqrt{1-V^2/c^2}}$. 用四式除前三式即得；只是用四维速度导出更简且具代表性）.

3). 四维动量

定义四维动量 $P_x = m_0 u_x$，$P_y = m_0 u_y$，$P_z = m_0 u_z$，$P_t = m_0 u_t$

$\therefore P = (P_x, P_y, P_z, P_t) = (\gamma m_0 v_x, \gamma m_0 v_y, \gamma m_0 v_z, i m_0 c\gamma)$

显然，四维动量服从洛伦兹变换：

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$$\begin{cases} P_x' = \dfrac{P_x + i\beta P_t}{\sqrt{1-V^2/C^2}} = \dfrac{P_x - \frac{VE}{C^2}}{\sqrt{1-V^2/C^2}} \\ P_y' = P_y \\ P_z' = P_z \\ P_t' = \dfrac{P_t - i\beta P_x}{\sqrt{1-V^2/C^2}} = \dfrac{P_t - i\frac{V}{C}P_x}{\sqrt{1-V^2/C^2}} \end{cases}$$

洛伦兹不变量：
$$P_x^2 + P_y^2 + P_z^2 + P_t^2 = \frac{m_0^2(U_x^2+U_y^2+U_z^2-C^2)}{1-U^2/C^2} = -m_0^2C^2.$$

⑤ 狭义相对论动力学. $U=(U_x, U_y, U_z, U_t)$：四维速度；$\vec{v}=(v_x, v_y, v_z)$：三维速度；
V：参考系相对速度.

1) 质量：$m = \dfrac{m_0}{\sqrt{1-V^2/C^2}}$

2) 动量：$\vec{P} = m\vec{v} = \dfrac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$；力：$\vec{F} = \dfrac{d\vec{P}}{dt} = \dfrac{d}{dt}\left(\dfrac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\right)$

3) 能量. $dE_k = \vec{F}d\vec{s} = \dfrac{d(m\vec{v})}{dt}\cdot d\vec{s} = v^2dm + mvdv$.

又∵ $m^2c^2 - m_0c^2 = m^2v^2$，∴ $2mc^2dm = 2mv^2dm + 2vm^2dv$

∴ $c^2dm = v^2dm + mvdv$ 代入前式，得

$dE_k = c^2dm$

∴ $E_k = mc^2 - m_0c^2$. ∴总能量 $E = mc^2$；功率：$p = \dfrac{dE_k}{dt} = \vec{F}\cdot\vec{v}$.

且有 $E^2 = m_0^2c^4 + p^2c^2$.

4) 力的变换公式：

定义四维力：$f_x = \dfrac{dP_x}{d\tau}$；$f_y = \dfrac{dP_y}{d\tau}$；$f_z = \dfrac{dP_z}{d\tau}$；$f_t = \dfrac{dP_t}{d\tau}$.

$$\therefore \begin{cases} f_x = \dfrac{dP_x}{dt}\dfrac{dt}{d\tau} = \dfrac{dP_x}{dt}\dfrac{1}{\sqrt{1-v^2/c^2}} = F_x/\sqrt{1-v^2/c^2}. \\ f_y = F_y/\sqrt{1-v^2/c^2},\quad f_z = F_z/\sqrt{1-v^2/c^2} \\ f_t = \dfrac{dP_t}{dt}\dfrac{dt}{d\tau} = \dfrac{im_0c\, d\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)}{dt}\dfrac{1}{\sqrt{1-v^2/c^2}} = \dfrac{im_0c}{\sqrt{1-v^2/c^2}\,(m_0c^2)}\cdot\dfrac{d\left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right)}{dt} = \dfrac{i\vec{F}\cdot\vec{v}}{c\sqrt{1-v^2/c^2}} \end{cases}$$

$$\therefore f_x' = \frac{f_x + i\beta f_t}{\sqrt{1-V^2/C^2}} = \frac{f_x - \frac{V}{C^2}\frac{\vec{F}\cdot\vec{v}}{\sqrt{1-v^2/c^2}}}{\sqrt{1-V^2/C^2}}$$

$$\therefore \frac{F_x'}{\sqrt{1-v'^2/c^2}} = \frac{\frac{F_x}{\sqrt{1-v^2/c^2}} - \frac{V}{c^2}\frac{\vec{F}\cdot\vec{v}}{\sqrt{1-v^2/c^2}}}{\sqrt{1-V^2/c^2}}$$ 根据 $\dfrac{\gamma_0\gamma}{\gamma'} = \dfrac{1}{1-\frac{Vv_x}{c^2}}$

$$\therefore F_x' = \frac{F_x - \frac{V}{c^2}\vec{F}\cdot\vec{v}}{1-\frac{Vv_x}{c^2}}$$

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又∵ $f_y' = f_y$

∴ $\frac{F_y'}{\sqrt{1-v'^2/c^2}} = \frac{F_y}{\sqrt{1-v^2/c^2}}$　∴ $F_y' = \frac{F_y\sqrt{1-V^2/C^2}}{1-\frac{Vv_x}{C^2}}$

同理 $F_z' = \frac{F_z\sqrt{1-V^2/C^2}}{1-\frac{Vv_x}{C^2}}$

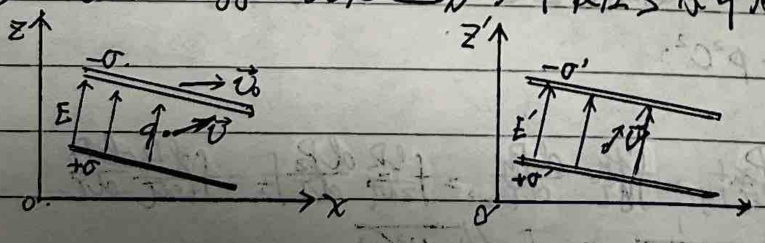
$f_t' = \frac{f_t - i\beta f_x}{\sqrt{1-V^2/C^2}}$ ……

综合上几式，有：Fx、Fy、Fz皆为三维力.

$$\begin{cases} F_x' = \dfrac{F_x - \frac{V}{c^2}\vec{F}\cdot\vec{v}}{1-\frac{Vv_x}{C^2}} \\ F_y' = \dfrac{F_y\sqrt{1-V^2/C^2}}{1-\frac{Vv_x}{C^2}} \\ F_z' = \dfrac{F_z\sqrt{1-V^2/C^2}}{1-\frac{Vv_x}{C^2}} \end{cases}$$

应用举例：磁场的来源：

带电平行平板在S系中以 $\vec{v}_0 = v_0\vec{i}$ 运动，板间有一电荷q以速度 $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ 运动；平板在S'系中静止，q在S'系中 $\vec{v}' = v_x'\vec{i} + v_y'\vec{j} + v_z'\vec{k}$



S'系中. $F_x' = qE_x' = q(\frac{\sigma'}{\varepsilon_0})_x = q(\frac{\sigma}{\varepsilon_0})_x = qE_x$　　$v_x' = \frac{v_x - v_0}{1-\frac{v_0v_x}{c^2}}$

$F_y' = qE_y' = q(\frac{\sigma'}{\varepsilon_0})_y = q(\frac{\sigma\sqrt{1-v_0^2/c^2}}{\varepsilon_0})_y = qE_y\sqrt{1-v_0^2/c^2}$　　$v_y' = \frac{v_y}{1-\frac{v_0v_x}{c^2}}\sqrt{1-v_0^2/c^2}$

$F_z' = qE_z' = q(\frac{\sigma'}{\varepsilon_0})_z = q(\frac{\sigma\sqrt{1-v_0^2/c^2}}{\varepsilon_0})_z = qE_z\sqrt{1-v_0^2/c^2}$　　$v_z' = \frac{v_z}{1-\frac{v_0v_x}{c^2}}\sqrt{1-v_0^2/c^2}$

根据力的变换公式，在S系中.

$$F_x = \frac{F_x' + \frac{v_0}{c^2}(F_x'v_x' + F_y'v_y' + F_z'v_z')}{1+\frac{v_x'v_0}{c^2}} = \frac{qE_x + \frac{qv_0}{c^2}\left[E_x\frac{v_x-v_0}{1-\frac{v_0v_x}{c^2}} + \left(\frac{E_yv_y}{1-\frac{v_0v_x}{c^2}} + \frac{E_zv_z}{1-\frac{v_0v_x}{c^2}}\right)(1-v_0^2/c^2)\right]}{1+\frac{v_xv_0 - v_0^2}{c^2-v_0v_x}}$$

$$= qE_x + q(E_y v_y + E_z v_z)\frac{v_0}{c^2}$$

$$F_y = \frac{qE_y\sqrt{1-v_0^2/c^2}\cdot\sqrt{1-v_0^2/c^2}}{1+\frac{v_x v_0 - v_0^2}{c^2 - v_0 v_x}} = qE_y - \frac{q v_0 v_x E_y}{c^2}$$

$$F_z = qE_z - \frac{q v_0 v_x E_z}{c^2}.$$

定义 $\vec{B} = \frac{1}{c^2}\vec{v_0}\times\vec{E}$.

$\therefore \vec{F} = \vec{F_x} + \vec{F_y} + \vec{F_z} = q(\vec{E_x} + \vec{E_y} + \vec{E_z}) + q\cdot\vec{v}\times\vec{B} = q\vec{E} + q\vec{v}\times\vec{B}$

⑥ 核反应和粒子反应

1) 原子核: $M = N + Z$ (M:质量数, N:中子数, Z:质子数)

- α衰变: ${}^{M}_{Z}X \to {}^{M-4}_{Z-2}Y + {}^{4}_{2}He$: 放出α粒子
- β衰变: ${}^{M}_{Z}X \to {}^{M}_{Z+1}Y + {}^{0}_{-1}e$: 放出β粒子

} 放出γ光子.

- $N = N_0 e^{-\lambda t}$, 半衰期 T: $N = \frac{N_0}{2}$, $\therefore e^{-\lambda t} = \frac{1}{2}$, $T = \frac{\ln 2}{\lambda}$
- $\therefore N = N_0 e^{t(-\frac{\ln 2}{T})} = N_0(e^{\ln\frac{1}{2}})^{\frac{t}{T}} = N_0(\frac{1}{2})^{\frac{t}{T}}$.

- 质子的发现: ${}^{14}_{7}N + {}^{4}_{2}He \to {}^{17}_{8}O + {}^{1}_{1}H$.
- 中子的发现: ${}^{9}_{4}Be + {}^{4}_{2}He \to {}^{12}_{6}C + {}^{1}_{0}n$
- 正电子的发现: ${}^{30}_{15}P \to {}^{30}_{14}Si + {}^{0}_{1}e$.

- 核裂变: ${}^{235}_{92}U + {}^{1}_{0}n \to {}^{144}_{56}Ba + {}^{92}_{36}Kr$
- 核聚变: ${}^{2}_{1}H + {}^{3}_{1}H \to {}^{4}_{2}He + {}^{1}_{0}n$.

2). 两体衰变: $A \to A_1 + A_2$

$$\begin{cases} P_1 = P_2 \\ m_0 c^2 = \sqrt{P_1^2 c^2 + m_{10}^2 c^4} + \sqrt{P_2^2 c^2 + m_{20}^2 c^4} \end{cases}$$

解得 $P_1 = P_2 = \frac{c}{2m_0}\sqrt{[m_0^2 - (m_{10}+m_{20})^2][m_0^2 - (m_{10}-m_{20})^2]}$

$$\begin{cases} E_1 = \sqrt{P_1^2 c^2 + m_{10}^2 c^4} = \frac{m_0^2 + m_{10}^2 - m_{20}^2}{2m_0}c^2 \\ E_2 = \sqrt{P_2^2 c^2 + m_{20}^2 c^4} = \frac{m_0^2 + m_{20}^2 - m_{10}^2}{2m_0}c^2. \end{cases}$$

3) 两体反应、阈能.

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$$A_1+A_2\longrightarrow A_3+A_4+A_5\cdots$$

在动心系（动量中心系，相对论中，动心系一般不与质心系重合）中：

<1> $P_1'=P_2'$

<2> $P_1'=\gamma(P_1-\beta\frac{E_1}{C})$ } 高能粒子

<3> $E_1'=\gamma(E_1-\beta P_1C)$

<4> $P_2'=\gamma\beta m_{20}C$ } 靶粒子（在实验室系中静止）

<5> $E_2'=\gamma m_{20}C^2$.

（由四维动量变换的最后一式可得 $E'=(E-VP_x)/\sqrt{1-V^2/C^2}$.）

由<1><2><4>式解得 $\beta=\frac{P_1C}{E_1+m_{20}C^2}$.

$$\therefore E_1'+E_2'=\gamma(E_1-\beta P_1C+m_{20}C^2)=\frac{1}{\sqrt{1-\beta^2}}\left(E_1-\frac{P_1^2C^2}{E_1+m_{20}C^2}+m_{20}C^2\right)$$

$$=\frac{1}{\sqrt{1-\frac{E_1^2-m_{10}^2C^4}{(E_1+m_{20}C^2)^2}}}\left(E_1-\frac{E_1^2-m_{10}^2C^4}{E_1+m_{20}C^2}+m_{20}C^2\right)$$

$$=\sqrt{m_{10}^2C^4+m_{20}^2C^4+2E_1m_{20}C^2}$$

这是动心系中的总能量 E' 的表达式。在动心系中，这部分能量可完全转化为生成物的静质能，即反应产物在动心系中全部静止，没有动能。故这个能量 E' 也叫反应有效能。

设发生这种阈能反应时，入射粒子 m_{10} 的能量阈值为 E_m，在动心系中的动能为 E_{km}.

$$\therefore\sqrt{m_{10}^2C^4+m_{20}^2C^4+2E_mm_{20}C^2}=\sum_{i=3}^{n}m_iC^2$$ （反应后所有生成物的静质能）

解得 $E_m=\frac{(\sum_{i=3}^{n}m_iC^2)^2-(m_{10}^2C^4+m_{20}^2C^4)}{2m_{20}C^2}$ $\therefore E_{km}=E_m-m_{10}C^2=$

$$\frac{(\sum_{i=3}^{n}m_iC^2)^2-(m_{10}C^2+m_{20}C^2)^2}{2m_{20}C^2}=\frac{(\sum_{i=3}^{n}m_i+m_{10}+m_{20})(\sum_{i=3}^{n}m_i-m_{10}-m_{20})C^2}{2m_{20}}$$

$$=\left[-(m_{10}+m_{20}-\sum_{i=3}^{n}m_i)C^2\right]\cdot\frac{\sum_{i=1}^{n}m_i}{2m_{20}}$$ [$Q=(m_{10}+m_{20}-\sum_{i=3}^{n}m_i)C^2$:反应能]

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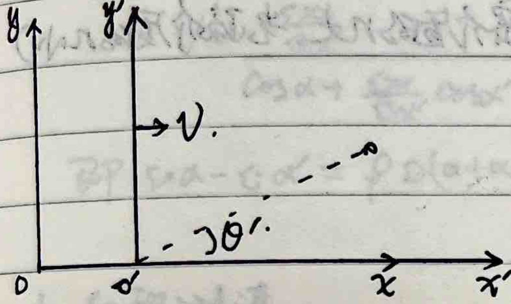
㊁ 多普勒效应.

设光源相对K系静止，观察者相对K′系静止，K′系以速度v沿K系x轴正向运动，K′系中光源到观察者的连线与x轴的夹角为θ'.



在K′系中光子沿x'轴的分量 $P_x' = p'\cos\theta'$

$= \frac{E'}{c}\cos\theta' = \frac{h\nu'}{c}\cos\theta'$.

由洛仑兹变换 $E = \frac{1}{\sqrt{1-\beta^2}}(E' - \beta c P_x')$

$\therefore h\nu = \frac{1}{\sqrt{1-\beta^2}}(h\nu' - h\nu'\beta\cos\theta')$

$\therefore \frac{\nu'}{\nu} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta'}$

$\begin{cases} \theta' = 0\text{时}, \frac{\nu'}{\nu} = \sqrt{\frac{1+\beta}{1-\beta}} \approx 1+\beta \\ \theta' = \frac{\pi}{2}\text{时}, \frac{\nu'}{\nu} = \sqrt{1-\beta^2} \approx 1-\frac{1}{2}\beta^2 \end{cases}$

三. 几何光学

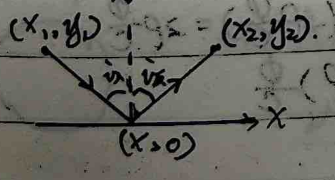
1. 光程：$\Delta = ct = n\nu t = ns$

光程差：$\delta = n_1 s_1 - n_2 s_2$.

(光速 $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$，$v = \frac{c}{\sqrt{\varepsilon_r\mu_r}}$，$n = \sqrt{\varepsilon_r\mu_r}$.)

费马原理：光沿最短或最长光程传播，$\int_a^b n\,ds = $ 极值.

2. 光的反射

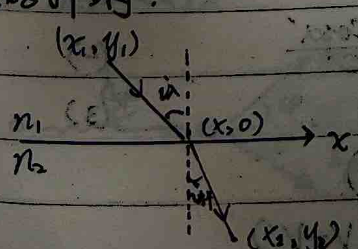


$\Delta = n\sqrt{(x_1-x)^2+y_1^2} + n\sqrt{(x_2-x)^2+y_2^2}$

$\frac{d\Delta}{dx} = n\cdot\frac{1}{2}\frac{2(x_1-x)(-dx)}{\sqrt{(x_1-x)^2+y_1^2}} + n\cdot\frac{1}{2}\frac{2(x_2-x)(-dx)}{\sqrt{(x_2-x)^2+y_2^2}} = 0.$

$\therefore \frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}} = \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}}$ $\therefore i_入 = i_反$.

3. 光的折射.



$\Delta = n_1\sqrt{(x_1-x)^2+y_1^2} + n_2\sqrt{(x_2-x)^2+y_2^2}$

$\frac{d\Delta}{dx} = n_1\frac{1}{2}\frac{-2(x_1-x)dx}{\sqrt{(x_1-x)^2+y_1^2}} + n_2\frac{1}{2}\frac{-(x_2-x)dx}{\sqrt{(x_2-x)^2+y_2^2}} = 0.$

$\therefore \frac{n_1(x-x_1)}{\sqrt{(x-x_1)^2+y_1^2}} = \frac{n_2(x_2-x)}{\sqrt{(x_2-x)^2+y_2^2}}$ $\therefore \sin i_入\cdot n_1 = \sin i_折\, n_2$.

4. 全反射

$\frac{\sin\theta}{\sin\alpha}=n$，当 $\sin\theta=1$，即折射角为 $90°$ 时，发生全反射。

此时 $\sin\alpha=\frac{1}{n}$，$\alpha=\arcsin\frac{1}{n}$（$n=\frac{c}{v}$，光密介质的 n 大，光疏介质的 n 小）

专题：光子

1. 理论基础：

$E=h\nu$，$p=\frac{h\nu}{c}=\frac{h}{\lambda}$。

$\because E^2=p^2c^2+m_0^2c^4$，而 $m_0=0$ $\therefore p=\frac{E}{c}$。

$$\because m=\frac{m_0}{\sqrt{1-v^2/c^2}}=\sqrt{\frac{\frac{E^2-p^2c^2}{c^4}}{\frac{c^2-v^2}{c^2}}}=\sqrt{\frac{E^2-p^2c^2}{c^2(c^2-v^2)}}=\sqrt{\frac{(h\nu)^2-p^2c^2}{c^2(c^2-v^2)}}$$

$$=\sqrt{\frac{\frac{h^2c^2}{\lambda^2}-m^2c^4}{c^2(c^2-v^2)}}=\sqrt{\frac{\frac{h^2}{\lambda^2}-m^2c^2}{c^2-v^2}}$$

$\therefore m^2(c^2-v^2)=\frac{h^2}{\lambda^2}-m^2c^2$ $\therefore m^2(2c^2-v^2)=\frac{h^2}{\lambda^2}$

$\because v=c$，$\therefore m=\frac{h}{c\lambda}=\frac{h\nu}{c^2}$，此为光子的动质量。

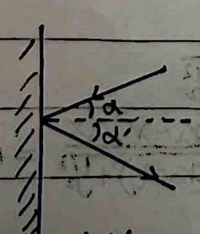
2. 光压：一群光子总动量 $p=\frac{Nh\nu}{c}=\frac{\Phi}{c}$

若壁的反射系数 ρ，则 $\Delta I_1=\rho\frac{\Phi}{c}-(-\rho\frac{\Phi}{c})=2\rho\frac{\Phi}{c}$。

余下 $(1-\rho)N$ 个光子被壁吸收，则 $\Delta I_2=(1-\rho)\frac{\Phi}{c}$

$\therefore$ 光压 $=\Delta I_1+\Delta I_2=(1+\rho)\frac{\Phi}{c}$.

3. 反射定律推广：



$$\begin{cases} h\nu+\frac{1}{2}MV^2=h\nu'+\frac{1}{2}MV'^2 & (1)\\ MV-\frac{h\nu\cos\alpha}{c}=MV'+\frac{h\nu'\cos\alpha'}{c} & (2)\\ \frac{h\nu\sin\alpha}{c}=\frac{h\nu'\sin\alpha'}{c} & (3)\end{cases}$$

由(1)得 $h(\nu'-\nu)=\frac{1}{2}M(V^2-V'^2)$

由(2)得 $\frac{h}{c}(\nu\cos\alpha+\nu'\cos\alpha')=M(V-V')$

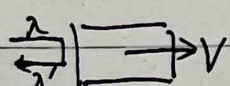
$$\therefore\ c\,\frac{\nu'-\nu}{\nu\cos\alpha+\nu'\cos\alpha'}=\frac{1}{2}(V+V')\approx V$$

由(3)得 $\nu'=\frac{\sin\alpha}{\sin\alpha'}\nu$，代入上式得：

$$V=\frac{\frac{\sin\alpha}{\sin\alpha'}-1}{\cos\alpha+\frac{\sin\alpha}{\sin\alpha'}\cos\alpha'}c=\frac{\sin\alpha-\sin\alpha'}{\sin\alpha'\cos\alpha+\sin\alpha\cos\alpha'}c=\frac{\sin\alpha-\sin\alpha'}{\sin(\alpha+\alpha')}c.$$

即 $\sin\alpha-\sin\alpha'=\beta\sin(\alpha+\alpha')$. 若$V$与镜成$\varphi$角，则 $\sin\alpha-\sin\alpha'=\beta\sin\varphi\sin(\alpha+\alpha')$.

4. 红移测速.



$$\begin{cases}\frac{h}{\lambda}+MV=-\frac{h}{\lambda'}+MV'\\ \frac{hc}{\lambda}+\frac{1}{2}MV^2=\frac{hc}{\lambda'}+\frac{1}{2}MV'^2\end{cases}$$

$$\therefore\ h\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right)=M(V'-V)$$

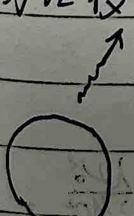
$$hc\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\frac{1}{2}M(V'^2-V^2)$$

$$\therefore\ c\,\frac{\frac{1}{\lambda}-\frac{1}{\lambda'}}{\frac{1}{\lambda}+\frac{1}{\lambda'}}=\frac{1}{2}(V'+V)\approx V$$

$$\therefore\ V=\frac{\lambda'-\lambda}{\lambda'+\lambda}\cdot c$$

又∵红移量 $\frac{\lambda'-\lambda}{\lambda}=z$ ∴ $V=\frac{z}{2+z}\cdot c$

5. 引力红移.

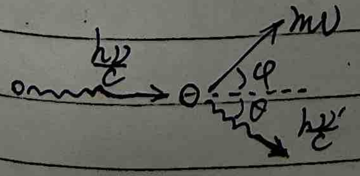


$$h\nu-\frac{GMm}{r}=h\nu'-\frac{GMm'}{r'}$$

$$\therefore\ h\nu\left(1-\frac{GM}{c^2r}\right)=h\nu'\left(1-\frac{GM}{c^2r'}\right)$$

$$\therefore\ \nu'=\frac{1-\frac{GM}{c^2r}}{1-\frac{GM}{c^2r'}}\cdot\nu.$$

6. 康普顿效应



由动量守恒和余弦定理：

$$(mv)^2=\left(\frac{h\nu}{c}\right)^2+\left(\frac{h\nu'}{c}\right)^2-2\frac{h\nu}{c}\frac{h\nu'}{c}\cdot\cos\theta.$$

由能量守恒：

$$m_0c^2+h\nu=mc^2+h\nu'.$$

由 $m=\frac{m_0}{\sqrt{1-v^2/c^2}}$，得 $m^2c^2=m^2v^2+m_0^2c^2$。∴ $(mc^2)^2=(mv)^2c^2+m_0^2c^4$.

将前两式代入此式。

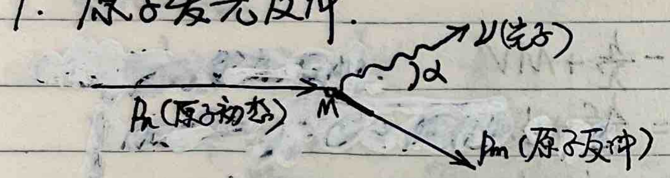
$$(m_0c^2+h\nu-h\nu')^2=\left[\left(\frac{h\nu}{c}\right)^2+\left(\frac{h\nu'}{c}\right)^2-\frac{2h^2\nu\nu'}{c^2}\cos\theta\right]c^2+m_0^2c^4.$$

展开，得：$m_0^2c^4+2m_0c^2h(\nu-\nu')+h^2(\nu-\nu')^2=h^2\nu^2+h^2\nu'^2-2h^2\nu\nu'\cos\theta+m_0^2c^4$

$$\therefore\ m_0c^2(\nu-\nu')=h\nu\nu'(1-\cos\theta)$$

$$\therefore\ \frac{\nu-\nu'}{\nu\nu'}\approx\frac{\Delta\nu}{\nu^2}=\frac{2h}{m_0c^2}\sin^2\frac{\theta}{2}.$$

7. 原子发光反冲.



忽略原子质量变化，由能量守恒和动量守恒，得：

$$\begin{cases}\frac{P_n^2}{2M}+E_n=\frac{P_m^2}{2M}+E_m+h\nu,\\ P_m^2=P_n^2+P^2-2P_nP\cos\alpha\end{cases}$$

$\therefore\ \nu=\frac{E_n-E_m}{h}+\frac{2P_nP}{2Mh}\cos\alpha-\frac{P^2}{2Mh}$　　∵ $P_n=Mv_n$，$P=\frac{h\nu}{c}$

$$\therefore\ \nu=\nu_0+\frac{v_n}{c}\nu\cos\alpha-\frac{h\nu^2}{2Mc^2}\approx\nu_0+\frac{v_n}{c}\nu_0\cos\alpha-\frac{h\nu_0^2}{2Mc^2}.$$

第一项为玻尔频率，

第二项为多普勒效应，当 $v_n=0$ 时，此项为零。$\nu=\nu_0-\frac{h\nu_0^2}{2Mc^2}$。

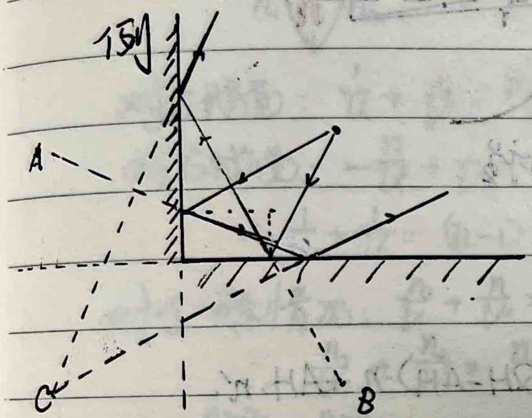
第三项为原子反冲的结果。

5. 反射镜成像.

①平面镜：

判断虚像个数：让物向每面镜发出一条光线，让光线在镜组间反射直至射出系统，每条反射光线的反向延长线必经过一虚像.

例

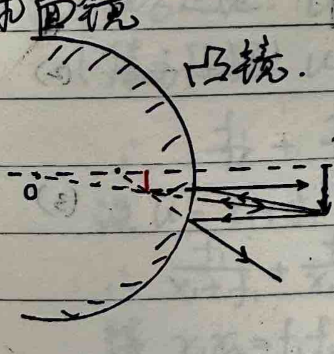


实像：实际光线汇聚成的像.

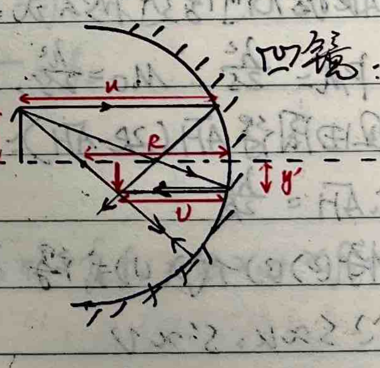
虚像：光线反向延长线汇聚成的像.

②球面镜

凸镜.



凹镜.



由图得 $\dfrac{u-R}{R-v}=\dfrac{y}{y'}\approx\dfrac{u-R+\frac{R}{2}}{\frac{R}{2}}=\dfrac{u-\frac{R}{2}}{\frac{R}{2}}$

得 $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{2}{R}=\dfrac{1}{f}$.（$f=\dfrac{R}{2}$：对近轴光线的近似）

符号法则：物在镜前（能反射光一侧），$u>0$；物在镜后，$u<0$.

实像 $v>0$；虚像 $v<0$.（未知时设为正，解出后看正负号）.

凹镜 $f>0$；凸镜 $f<0$.

放大率：设倒立的像高为负.

$\therefore \dfrac{-y'}{y}=\dfrac{v-R}{u-R}=\dfrac{v-\frac{2uv}{u+v}}{u-\frac{2uv}{u+v}}=\dfrac{v^2-uv}{u^2-uv}=-\dfrac{v}{u}$.

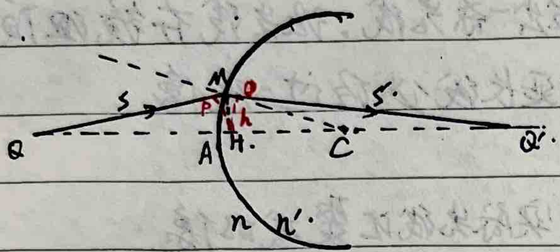
即 $K=-\dfrac{v}{u}$.

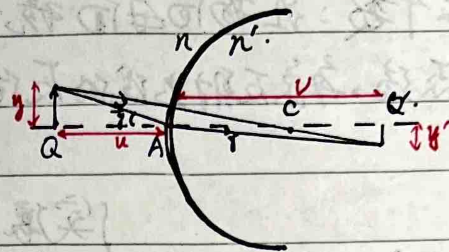
No.

Date

6. 透镜成像.

① 单球面折射.





根据费马原理，光路QMQ'与QAHCQ'等光程.

$\therefore QM\cdot n+MQ'\cdot n'=QA\cdot n+AQ'\cdot n'$

$\because QH=QP$，$Q'H=Q'O$.

$\therefore QP\cdot n+PM\cdot n+MO\cdot n'=QA\cdot n+AH\cdot n'=(QH-AH)\cdot n+AH\cdot n'$.

$\therefore PM\cdot n+MO\cdot n'=AH(n'-n)$ (1)

根据几何学近似公式，对于近轴光线QMQ'有

$PM=\frac{h^2}{2s}$，$MO=\frac{h^2}{2s'}$， (2)

又由图得 $\overline{AH}(2R-\overline{AH})=h^2\approx\overline{AH}\cdot 2R$

$\therefore \overline{AH}=\frac{h^2}{2R}$ (3)

将(2)(3)代入(1)式得 $\frac{h^2}{2s}\cdot n+\frac{h^2}{2s'}\cdot n'=\frac{h^2}{2R}(n'-n)$

$\because s\approx u$，$s'\approx v$

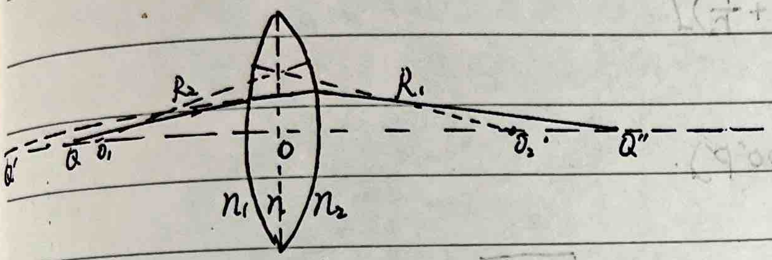
$\therefore \frac{n}{u}+\frac{n'}{v}=\frac{n'-n}{R}$

焦距：
- 物方焦距：$v\to\infty$时，$u=\frac{Rn}{n'-n}=f_1$；
- 像方焦距：$u\to\infty$时，$v=\frac{Rn'}{n'-n}=f_2$；

符号法则：公式中$n'-n$的顺序不变，只有u、v、R的符号变
- Q(物)在顶点A左，$u>0$，否则为负.
- Q'(像)在顶点A右，$v>0$，否则为负.
- C(球心)在A右，规定$R>0$，反之为负.

放大率：$K=\frac{y'}{y}=-\frac{v-r}{u+r}=-\frac{vn}{un'}$.

② 薄透镜成像：



对左球面：$\frac{1}{u}+\frac{n}{v'}=\frac{n-1}{R_1}$

对右球面：$-\frac{n}{v'}+\frac{1}{v}=\frac{1-n}{-R_2}$

∴ $\frac{1}{u}+\frac{1}{v}=(n-1)(\frac{1}{R_1}+\frac{1}{R_2})$，$f=\frac{1}{(n-1)(\frac{1}{R_1}+\frac{1}{R_2})}$

对于一般情况：$\frac{n_1}{u}+\frac{n}{v'}=\frac{n-n_1}{R_1}$，$-\frac{n}{v'}+\frac{n_2}{v}=\frac{n_2-n}{-R_2}$

∴ $\frac{n_1}{u}+\frac{n_2}{v}=\frac{n-n_1}{R_1}+\frac{n-n_2}{R_2}$.

当 $\frac{n-n_1}{R_1}+\frac{n-n_2}{R_2}>0$时，为会聚透镜，反之为发散透镜。

焦距：物方焦距：$v\to\infty$时，$u=\frac{n_1}{\frac{n-n_1}{R_1}+\frac{n-n_2}{R_2}}=f_1$

像方焦距：$u\to\infty$时，$v=\frac{n_2}{\frac{n-n_1}{R_1}+\frac{n-n_2}{R_2}}=f_2$

∴ $\frac{f_1}{u}+\frac{f_2}{v}=1$

设 $u=x_1+f_1$，$v=x_2+f_2$

∴ $\frac{f_1}{x_1+f_1}+\frac{f_2}{x_2+f_2}=1$　　∴ $f_1x_2+f_1f_2+f_2x_1+f_2f_1=x_1x_2+x_1f_2+f_1x_2+f_1f_2$

得 $x_1x_2=f_1f_2$

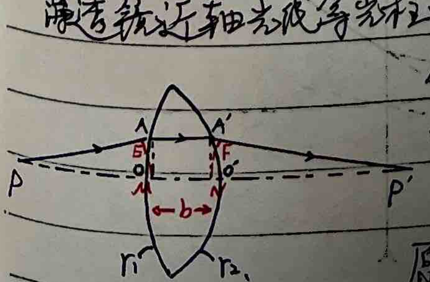
符号法则：实物 $u>0$，虚物 $u<0$

实像 $v>0$，虚像 $v<0$

凸透镜 $f>0$，凹透镜 $f<0$.

放大率：$K=K_1\cdot K_2=(-\frac{v'}{u}\frac{1}{n})\cdot(-\frac{v}{-v'}n)=-\frac{v}{u}$.

薄透镜近轴光线等光程的证明：



$\Delta(PAA'P')-\Delta(POO'P')=(PA+n\cdot AA'+A'P')-(PO+n\cdot OO'+O'P')$

$=(\cancel{PE}+EA+n\cdot\cancel{AA'}+A'F+\cancel{FP'})-[\cancel{PM}-OM+n(OM+\cancel{MN}+NO')+\cancel{NP'}-NO']$

∵ $AA'\approx MN=b$，$PE=PM$，$P'F=P'N$

原式$=EA+A'F+OM+NO'-n\cdot OM-n\cdot NO'$

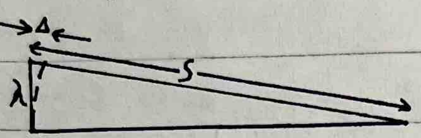
$$\approx \frac{h^2}{2\cdot PA}+\frac{h^2}{2P'A'}+\frac{h^2}{2r_1}+\frac{h^2}{2r_2}-n\frac{h^2}{2r_1}-n\frac{h^2}{2r_2}$$

$$=\frac{h^2}{2}\left[\frac{1}{u}+\frac{1}{v}+(1-n)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]$$

$$=\frac{h^2}{2}\left(\frac{1}{u}+\frac{1}{v}-\frac{1}{f}\right)$$

$$=0 \quad \therefore \Delta(PAA'P')=\Delta(POO'P').$$

($\Delta\approx\frac{h^2}{2s}$ 的证明



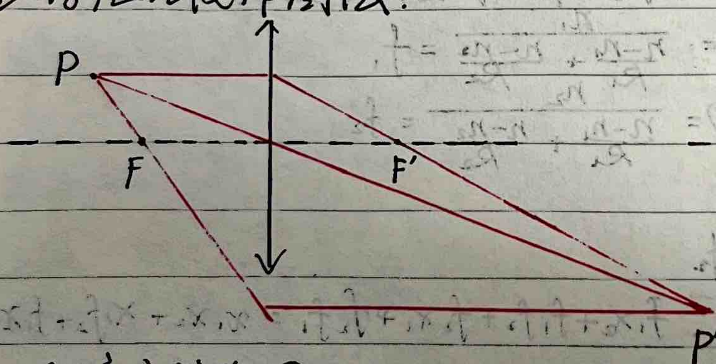
$\because \sqrt{s^2-h^2}=s-\Delta$

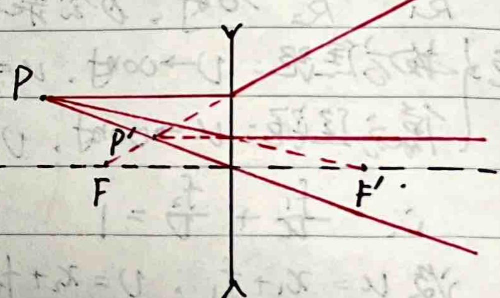
$\therefore s^2-h^2=s^2-2\Delta s+\Delta^2$

$\because \Delta^2\approx 0 \quad \therefore \Delta=\frac{h^2}{2s}$)

③ 作图法：

1) 特征光线作图法：

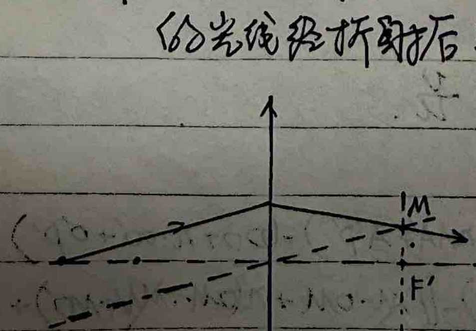


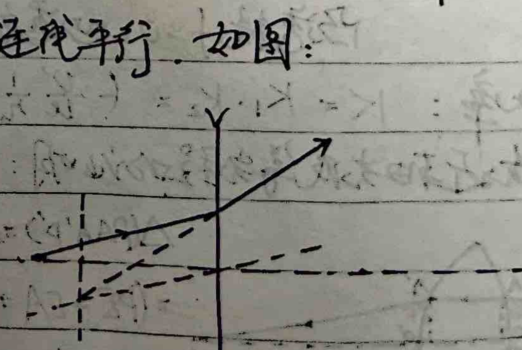


2) 任意光线作图法：

平行于副光轴的光线经透镜折射后通过焦平面与副光轴交点。

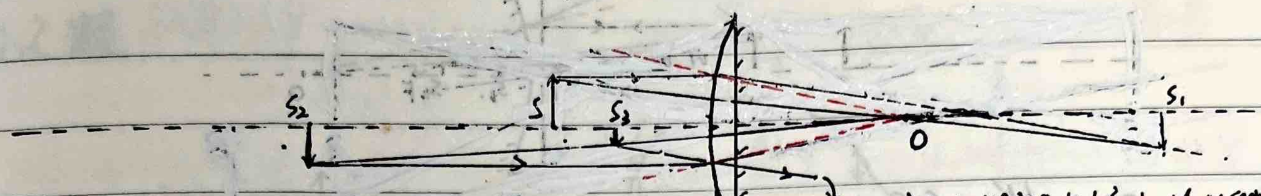
证明：将M看成一实物，由于它在焦平面上，它必成不了像，即由它发出的光线经折射后与它和光心的连线平行。如图：





专题：镀银透镜.

1. 凸平薄透镜（半径R，折射率n）



（此时S_2仍相当于在镜中，故折射角>入射角）

$\frac{1}{u}+\frac{n}{v}=\frac{n-1}{R}$，经反射 $u'=-v$ ∴ $-\frac{n}{v}+\frac{1}{v'}=\frac{1-n}{-R}$.（正方向反向）

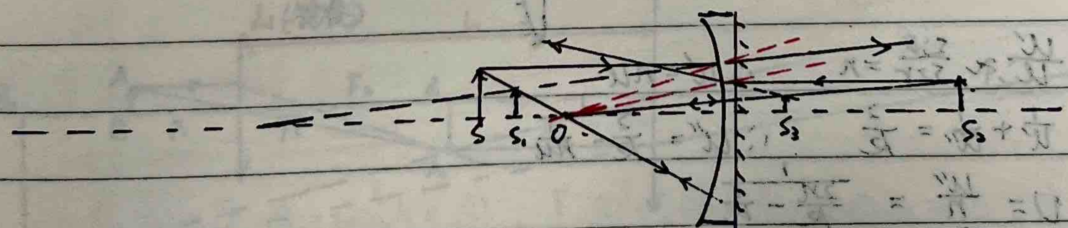
解得 $v'=\dfrac{1}{-\frac{1}{u}+\frac{2(n-1)}{R}}$.

放大率 $k=(-\frac{v}{u}\cdot\frac{1}{n})\cdot(-\frac{v'}{v}\cdot n)=-\frac{v'}{u}$.

焦距：当平行光射入时：

$$\left.\begin{cases}\frac{1}{\infty}+\frac{n}{v}=\frac{n-1}{R}\\ -\frac{n}{v}+\frac{1}{f}=\frac{1-n}{-R}\end{cases}\right\} f=\frac{R}{2(n-1)}.$$

2. 凹平薄透镜（半径R，折射率n）



$\frac{1}{u}+\frac{n}{v}=\frac{n-1}{-R}$，经反射 $u'=-v$. ∴ $-\frac{n}{v}+\frac{1}{v'}=\frac{1-n}{R}$（正方向反向）

解得 $v'=\dfrac{1}{-\frac{1}{u}-\frac{2(n-1)}{R}}$...

放大率 $k=(-\frac{v}{u}\cdot\frac{1}{n})(-\frac{v'}{v}\cdot n)=-\frac{v'}{u}$.

焦距：当平行光射入时：

$$\left.\begin{cases}\frac{1}{\infty}+\frac{n}{v}=\frac{n-1}{-R}\\ -\frac{n}{v}+\frac{1}{f}=\frac{1-n}{R}\end{cases}\right\} f=-\frac{R}{2(n-1)}$$

3. 平凸薄透镜（半径R，折射率n）

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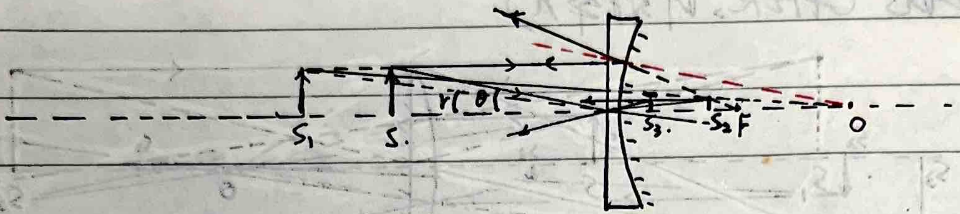


∵ $\frac{u'}{u} \approx \frac{\sin\theta}{\sin r} = n$ ∴ $u' = nu$.

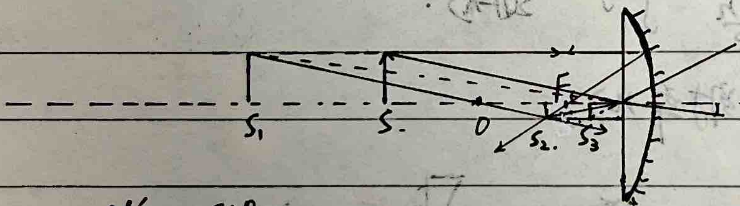
又∵ $\frac{1}{u'} + \frac{1}{u''} = -\frac{2}{R}$ ∴ $u'' = \frac{1}{-\frac{2}{R} - \frac{1}{nu}}$

∴ $v = \frac{u''}{n} = -\frac{1}{\frac{2n}{R} - \frac{1}{u}}$

焦距：$-\frac{\frac{R}{2}}{f} \approx n$，∴ $f = -\frac{R}{2n}$.

放大率：$K = -\frac{u''}{u'} = \frac{1}{\frac{2}{R} + \frac{1}{nu}} \cdot \frac{1}{nu} = \frac{1}{\frac{2nu}{R} + 1}$

4. 平凹薄透镜（半径R，折射率n）



（此时S₂仍视为于左镜中故 折射角>入射角）.

∵ $\frac{u'}{u} \approx \frac{\sin\theta}{\sin r} = n$ ∴ $u' = nu$

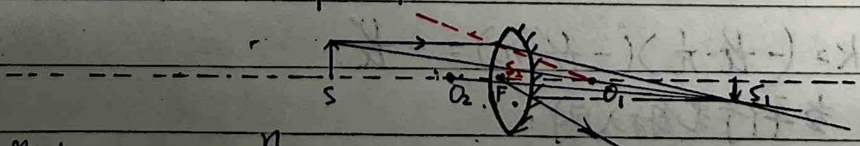
又∵ $\frac{1}{u'} + \frac{1}{u''} = \frac{2}{R}$ ∴ $u'' = \frac{1}{\frac{2}{R} - \frac{1}{nu}}$

∴ $v = \frac{u''}{n} = \frac{1}{\frac{2n}{R} - \frac{1}{u}}$

焦距：$\frac{\frac{R}{2}}{f} = n$，∴ $f = \frac{R}{2n}$

放大率：$K = -\frac{u''}{u'} = \frac{1}{\frac{2nu}{R} - 1}$

5. 凸凹薄透镜（半径R_1、R_2，折射率n）......



$\frac{1}{u} + \frac{n}{v_1} = \frac{n-1}{R_1}$，∴ $v_1 = \frac{n}{\frac{n-1}{R_1} - \frac{1}{u}}$，∴ $u_2 = -v_1$，$\frac{1}{u_2} + \frac{1}{v_2} = \frac{2}{R_2}$

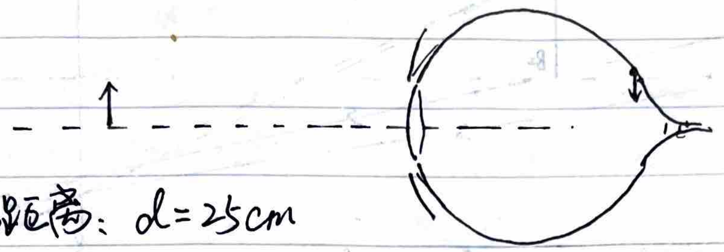
∴ $v_2 = \frac{1}{\frac{2}{R_2} + \frac{1}{v_1}} = \frac{1}{\frac{2}{R_2} + \frac{n-1}{nR_1} - \frac{1}{nu}}$ ∴ $u_3 = -v_2$，$\frac{n}{u_3} + \frac{1}{v_3} = \frac{1-n}{-R_1}$（正方向反向）.

∴ $v_3 = \frac{1}{\frac{2(n-1)}{R_1} + \frac{2n}{R_2} - \frac{1}{u}}$ 焦距：$u \to \infty$时，$f = \frac{1}{\frac{2(n-1)}{R_1} + \frac{2n}{R_2}}$，

放大率：$K = \left(-\frac{v_1}{u}\cdot\frac{1}{n}\right)\left(-\frac{v_2}{u_2}\right)\left(-\frac{v_3}{u_3}n\right) = -\frac{v_3}{u}$.

7. 光学仪器.

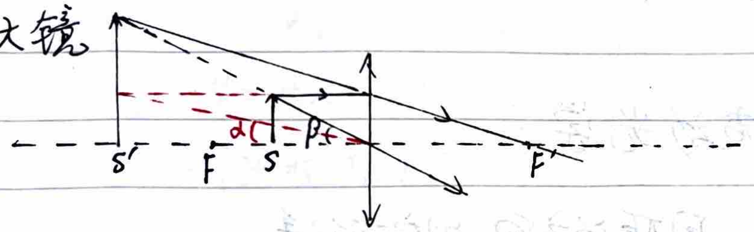
① 人眼



明视距离：$d=25cm$

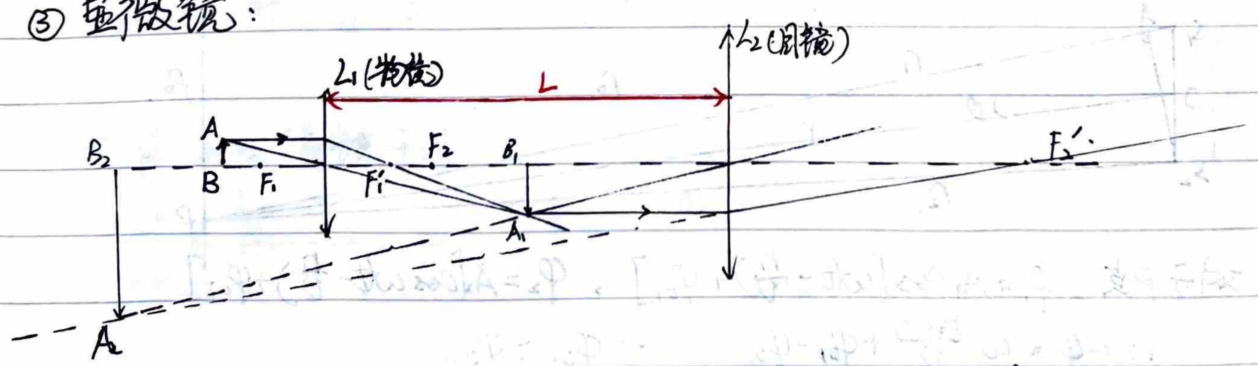
即眼能看清像的最小距离，为一般光学仪器中像到眼或目镜的距离.

② 放大镜



$$K=\frac{v}{u}\approx\frac{\beta}{\alpha}\approx\frac{h/d}{h/d}\approx\frac{h/f}{h/d}=\frac{d}{f}.$$ （d：明视距离，f：焦距）.

③ 显微镜：



$$K=\frac{\beta}{\alpha}\approx\frac{A_2B_2/d}{AB/d}\approx\frac{A_1B_1/f_2}{AB/d}=\frac{v_1}{u_1}\cdot\frac{d}{f_2}\approx\frac{L}{f_1}\cdot\frac{d}{f_2}=\frac{Ld}{f_1f_2}.$$

④ 开普勒望远镜.

$$K=\frac{\beta}{\alpha}\approx\frac{A_2B_2/d}{A_1B_1/f_1}\approx\frac{A_1B_1/f_2}{A_1B_1/f_1}=\frac{f_1}{f_2}.$$ ∴目镜f_2越小，物镜f_1越大，K越大.

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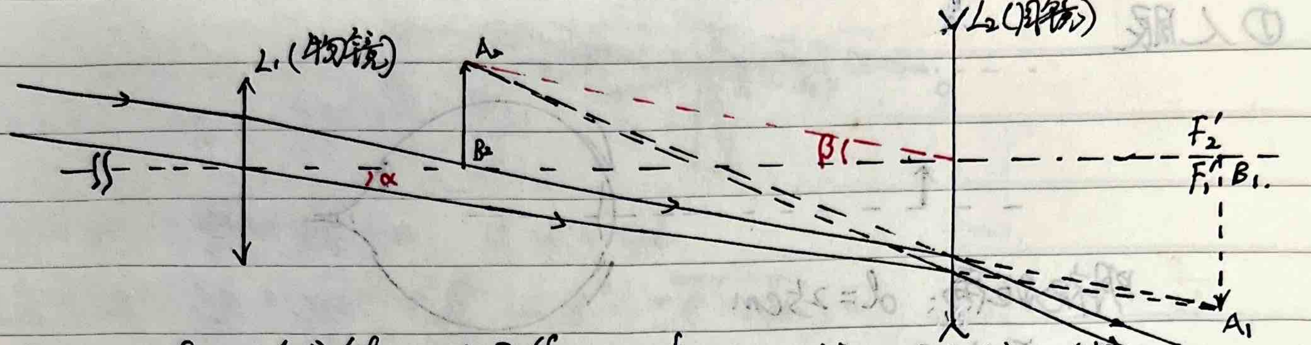
⑤ 伽利略望远镜.



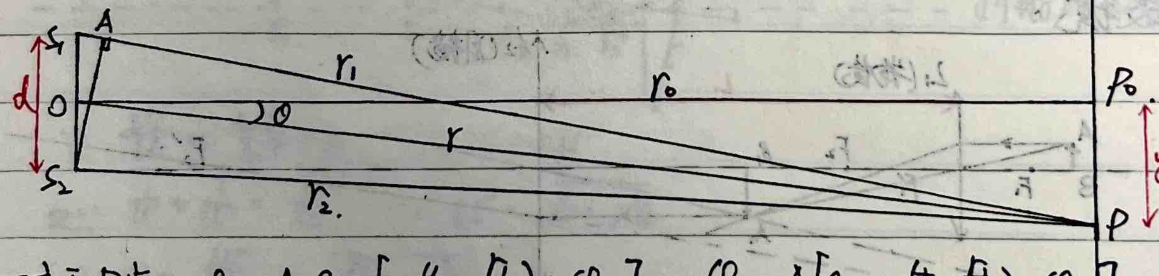
$K=\frac{\beta}{\alpha}\approx\frac{A_2B_2/d}{A_1B_1/f_1}\approx\frac{A_1B_1/f_2}{A_1B_1/f_1}=\frac{f_1}{f_2}$ ∴目镜(凹)f_2越小K越大.

四.波动光学

1.光的干涉：同频率.同振动方向、固定相位差.

①杨氏双缝干涉：

1)



对于P点，$\varphi_1=A_1\cos[\omega(t-\frac{r_1}{v})+\varphi_{01}]$，$\varphi_2=A_2[\cos\omega(t-\frac{r_2}{v})+\varphi_{02}]$

$\therefore\Delta\varphi=\omega\frac{r_2-r_1}{v}+\varphi_{01}-\varphi_{02}$　$\because\varphi_{01}=\varphi_{02}$

$\therefore\Delta\varphi=\frac{\omega}{v}(r_2-r_1)=\frac{2\pi}{\lambda}(r_2-r_1)=\frac{2\pi}{\lambda}\delta$.

$\therefore\delta=\frac{\Delta\varphi}{2\pi}\cdot\lambda$

若$\delta=k\lambda$.即$\frac{\Delta\varphi}{2\pi}=k$时，干涉相长

若$\delta=\frac{2k+1}{2}\lambda$.即$\frac{\Delta\varphi}{2\pi}=\frac{2k+1}{2}$时，干涉相消

又∵$\delta\approx AS_1=d\sin\theta\approx d\,tg\theta=d\frac{y}{r_0}$.

∴ 若$d\frac{y}{r_0}=k\lambda$，即$y=(2k)\frac{r_0}{d}\frac{\lambda}{2}$时，干涉相长

若$d\frac{y}{r_0}=\frac{2k+1}{2}\lambda$，即$y=(2k+1)\frac{r_0}{d}\frac{\lambda}{2}$时，干涉相消.

∴明暗条纹间距$\Delta y=\frac{r_0}{d}\lambda$.

2) 精确解：

若P点干涉相长，即振动最强，则P到两点光源 S_1、S_2 的距离差

$$\delta = 2k\cdot\frac{\lambda}{2} \quad (k=0,1,2,\cdots)$$

若k一定，则δ是一定值，到两定点距离差是定值的点的集合是双曲线，故两光源为双曲线两焦点。以 S_1S_2 所在直线为y轴，OP_0 所在直线为x轴。

$\because 2a=\delta=2k\cdot\frac{\lambda}{2}$，$\therefore a=k\cdot\frac{\lambda}{2}$，又 $\because c=\frac{d}{2}$

$$\therefore \frac{y^2}{(k\frac{\lambda}{2})^2}-\frac{x^2}{(\frac{d}{2})^2-(k\frac{\lambda}{2})^2}=1.$$

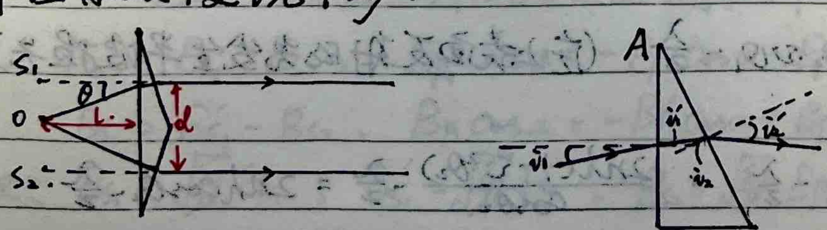
当 $x=r_0$ 时，$y=\pm k\cdot\frac{\lambda}{2}\sqrt{1+\frac{r_0^2}{(\frac{d}{2})^2-(k\frac{\lambda}{2})^2}}$

同理，若P点干涉相消，即振动最弱，有

$$\frac{y^2}{(\frac{2k+1}{2}\frac{\lambda}{2})^2}-\frac{x^2}{(\frac{d}{2})^2-(\frac{2k+1}{2}\frac{\lambda}{2})^2}=1.$$

当 $x=r_0$ 时，$y=\pm\frac{2k+1}{2}\frac{\lambda}{2}\sqrt{1+\frac{r_0^2}{(\frac{d}{2})^2-(\frac{2k+1}{2}\frac{\lambda}{2})^2}}$

② 菲涅尔双棱镜干涉：



取半部作为研究对象：$ni_1'\approx i_1$，$ni_2\approx i_2'$；$i_1'+i_2=A$。

$\therefore \theta=(i_1-i_1')+(i_2'-i_2)=(n-1)i_1'+(n-1)i_2=(n-1)A$

又 $\because \theta\approx\frac{d}{2}/l$，$\therefore d=2l(n-1)A$

$$\therefore\begin{cases}2l(n-1)A\cdot\frac{y}{l+r}=k\lambda \text{ 时，干涉相长} \\ 2l(n-1)A\,\frac{y}{l+r}=\frac{2k+1}{2}\lambda \text{ 时，干涉相消}\end{cases}$$

$$\Delta y=\frac{l+r}{2l(n-1)A}\cdot\lambda.$$

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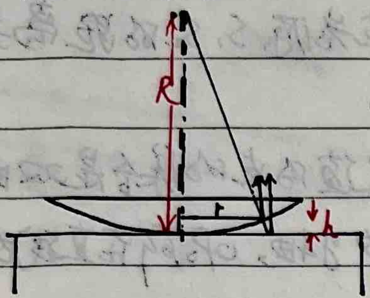
③ 牛顿环干涉.



$$\delta = 2h - \frac{\lambda}{2}$$

由于透过凸透镜而在平玻璃板上发生反射的光发生半波损失，光程差减去$\frac{\lambda}{2}$.

$\therefore 2h - \frac{\lambda}{2} = 2k\cdot\frac{\lambda}{2}$, $h = (2k+1)\frac{\lambda}{4}$时，干涉相长

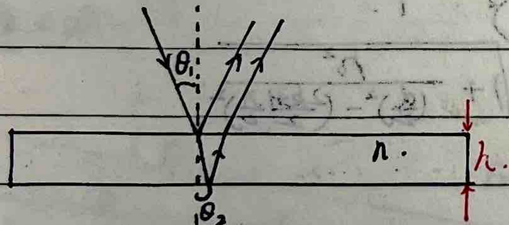
此时 $r^2 = h(2R-h) \approx 2Rh$, $r = \sqrt{2R(2k+1)\frac{\lambda}{4}}$

$2h - \frac{\lambda}{2} = (2k-1)\frac{\lambda}{2}$, $h = (2k)\frac{\lambda}{4}$时，干涉相消

此时 $r = \sqrt{2R\cdot 2k\cdot\frac{\lambda}{4}} = \sqrt{Rk\lambda}$.

④ 薄膜干涉.

1) 等倾干涉.（非平行光斜入射平玻璃）



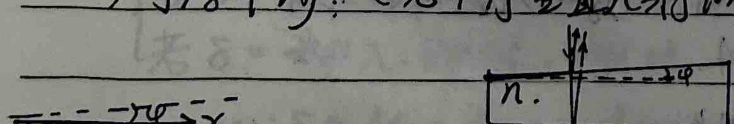
$$\delta = n\cdot\frac{2h}{\cos\theta_2} - 2\cdot h\,tg\theta_2\cdot\sin\theta_1 - \frac{\lambda}{2}$$（在上表面反射的光发生半波损失）

$\because \sin\theta_1 = n\sin\theta_2$

$$\therefore \delta = \frac{2nh - 2h\sin\theta_2\sin\theta_1}{\cos\theta_2} - \frac{\lambda}{2} = \frac{2nh(1-\sin^2\theta_2)}{\cos\theta_2} - \frac{\lambda}{2} = 2nh\cos\theta_2 - \frac{\lambda}{2}$$

$\therefore \begin{cases} 2nh\cos\theta_2 - \frac{\lambda}{2} = 2k\cdot\frac{\lambda}{2}, \text{即 } 2nh\cos\theta_2 = (2k+1)\frac{\lambda}{2}\text{ 时，干涉相长.} \\ 2nh\cos\theta_2 - \frac{\lambda}{2} = (2k-1)\frac{\lambda}{2}, \text{即 } 2nh\cos\theta_2 = 2k\frac{\lambda}{2}\text{ 时，干涉相消.} \end{cases}$

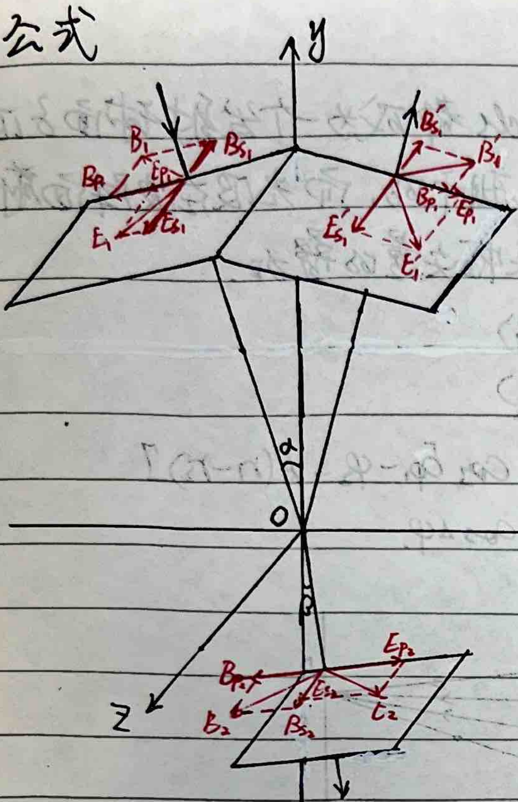
2) 等厚干涉.（光平行垂直入射倾斜小角度的玻璃）



$\begin{cases} h = (2k+1)\frac{\lambda}{4n\cos\theta_2} \approx (2k+1)\frac{\lambda}{4n}\text{ 时. 干涉相长} \\ h = 2k\cdot\frac{\lambda}{4n\cos\theta_2} \approx 2k\cdot\frac{\lambda}{4n}\text{ 时. 干涉相消.} \end{cases}$

$\therefore \Delta h = \frac{\lambda}{4n}$. $\Delta x = \frac{\Delta h}{\sin\varphi} \approx \frac{\Delta h}{\varphi} = \frac{\lambda}{4n\varphi}$. 故相邻明纹间距 $2\Delta x = \frac{\lambda}{2n\varphi}$.

⑤菲涅耳公式



入射光、反射光、折射光的速度$\vec{v}$、电矢量$\vec{E}$与磁矢量$\vec{B}$成右手螺旋关系。
图中$\vec{v_1}$、$\vec{v_1'}$、$\vec{v_2}$已定，故任给一组$\vec{E_1}$、$\vec{E_1'}$、$\vec{E_2}$，$\vec{B_1}$、$\vec{B_1'}$、$\vec{B_2}$就可给出，将六个量正交分解。
由各分量的连续性（在临界面处发生反射和折射后不会突变），得：

$$E_{s_1}=E_{s_1}'+E_{s_2},\quad E_{p_1}\cos\alpha=-E_{p_1}'\cos\alpha-E_{p_2}\cos\beta. \qquad (1)$$

$$B_{s_1}=B_{s_1}'-B_{s_2},\quad B_{p_1}\cos\alpha=-B_{p_1}'\cos\alpha+B_{p_2}\cos\beta. \qquad (2)$$

又∵ $B_{s_1}=\sqrt{\frac{\varepsilon_1}{\mu_1}}E_{p_1}$，$B_{s_1}'=\sqrt{\frac{\varepsilon_1}{\mu_1}}E_{p_1}'$，$B_{s_2}=\sqrt{\frac{\varepsilon_2}{\mu_2}}E_{p_2}$. 对于光波，$\mu_1=\mu_2=\mu_0$.

∴ $\sqrt{\varepsilon_1}E_{p_1}=\sqrt{\varepsilon_1}E_{p_1}'-\sqrt{\varepsilon_2}E_{p_2}$ ∴ $n_1E_{p_1}=n_1E_{p_1}'-n_2E_{p_2}$.

又∵ $n_1\sin\alpha=n_2\sin\beta$ ∴ $E_{p_1}=E_{p_1}'-\frac{\sin\alpha}{\sin\beta}E_{p_2}$. (3).

联立(1)、(3)得 $E_{p_1}\cos\alpha=-\cos\alpha\left(E_{p_1}+\frac{\sin\alpha}{\sin\beta}E_{p_2}\right)-E_{p_2}\cos\beta$.

$$\therefore \frac{E_{p_2}}{E_{p_1}}=-\frac{2\cos\alpha\sin\beta}{\sin\alpha\cos\alpha+\sin\beta\cos\beta}=-\frac{2\cos\alpha\sin\beta}{\frac{1}{2}(\sin2\alpha+\sin2\beta)}=\frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)\cos(\alpha-\beta)}.$$

同理，$E_{p_1}\cos\alpha=-E_{p_1}'\cos\alpha-\frac{\sin\beta}{\sin\alpha}(E_{p_1}'-E_{p_1})\cos\beta$

$$\therefore \frac{E_{p_1}'}{E_{p_1}}=-\frac{\sin\alpha\cos\alpha-\sin\beta\cos\beta}{\sin\alpha\cos\alpha+\sin\beta\cos\beta}=\frac{tg(\alpha-\beta)}{tg(\alpha+\beta)}.$$

还可证：$\frac{E_{s_1}'}{E_{s_1}}=\frac{-\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$

$\frac{E_{s_2}}{E_{s_1}}=\frac{2\sin\beta\cos\alpha}{\sin(\alpha+\beta)}$. 由此可解释光疏射向光密的光反射产生的半波损失.

2. 光的衍射:

由光源发出的光，其波阵面上每个ds都成为一个发射球面子波的波源，对局部波阵面来说，这些子波是相干的，而光波在波阵面前方某点的振幅矢量是这些子波在该点振幅矢量的叠加。

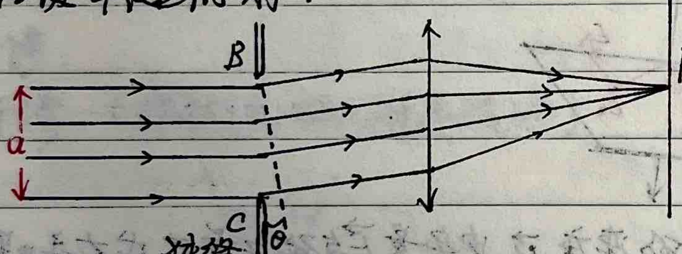
$$E_1 = E_{10}\cos(\omega t - kr_1 + \varphi_1)$$

$$E_2 = E_{20}\cos(\omega t - kr_2 + \varphi_2)$$

$$\therefore E_0^2 = E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos[\varphi_1 - \varphi_2 - k(r_1 - r_2)]$$

$\because I \propto E^2 \quad \therefore I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\varphi$.

① 夫琅和费单缝衍射.



狭缝

P点光强 $I = I_0\left(\frac{\sin\beta}{\beta}\right)^2$，$\beta = \frac{\pi}{\lambda}a\sin\theta \Rightarrow \sin\theta = \frac{\beta}{\pi}\cdot\frac{\lambda}{a}$.（可由惠更斯菲涅耳原理）

明纹：$\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right)^2 = 0$，得 $\tan\beta = \beta$，

解得 $\sin\theta = 0$，$\sin\theta = \pm1.43\frac{\lambda}{a}$，$\pm2.46\frac{\lambda}{a}$，$\pm3.47\frac{\lambda}{a}\cdots$

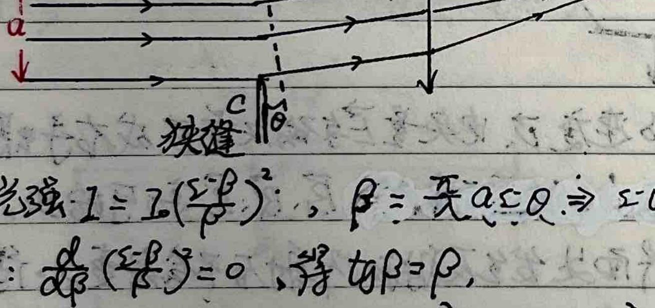
暗纹：$\beta = \pm n\pi$ 时，有 $I = 0$；

解得 $\sin\theta = \pm n\frac{\lambda}{a}$

定义中央明纹宽为两侧一级暗纹中心线间距，则 $\Delta x \approx 2(f\cdot\theta) \approx 2f\cdot\frac{\lambda}{a}$.（$n=1$）

② 光栅衍射.（大量等宽等间距平行狭缝构成的光学元件称为光栅）.

光栅常数
$d = a + b$

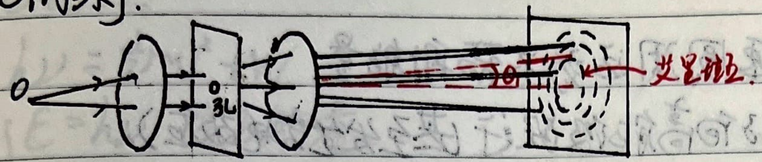


主极大：

$d\sin\theta = \pm k\lambda$.

（即 $\delta = 2k\cdot\frac{\lambda}{2}$）.

③圆孔衍射.



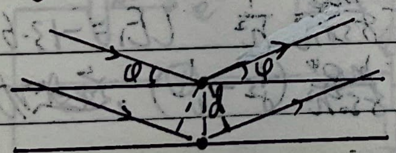
$d\sin\theta_1 = 1.22\lambda \Rightarrow \theta_1 \approx 1.22\frac{\lambda}{d}$.

按几何光学的观点，只要透镜有适当焦距总能把任何微小物体放大到任何程度，但事实并非如此！因为一个物点S发出的光经过透镜(圆孔)，它的像不再是一个点，而是艾里斑.

故人眼能分辨的两物点的最小间距 $\Delta l = d\cdot\theta_1 = d\cdot 1.22\frac{\lambda}{x}$

（d：明视距离25cm，x：瞳孔直径2-3mm，λ：入射光波长）

④ X射线的衍射：通过X射线在晶格中的衍射研究晶体内部结构.

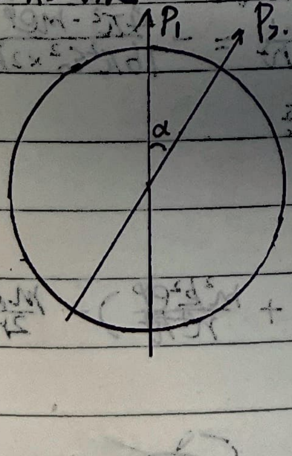


$\delta = 2d\sin\varphi$

$\therefore 2d\sin\varphi = k\lambda$. 干涉相长.

——布拉格公式.

3. 光的偏振.



$I_1 = \sqrt{\frac{\varepsilon}{\mu_0}}\sum_i E_{i1}^2$

$I_2 = \sqrt{\frac{\varepsilon}{\mu_0}}\sum_i (E_{i1}\cos\alpha)^2$.

$\therefore I_2 = I_1\cos^2\alpha$.

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五. 量子光学.

1. 波尔原子理论.

①假设：1)电子做匀速圆周运动不辐射能量.

2)电子吸收光子向高能级跃迁(甚至发生光电效应),

电子辐射光子向低能级跃迁（基态：$n=1$).

3) $mvr = n\frac{h}{2\pi}$, $n=1,2,3\cdots$.

②推论与推导：

$r=\frac{\varepsilon_0 h^2}{\pi m e^2}n^2 = n^2 r_1$, $v=\frac{e^2}{2\varepsilon_0 h}\frac{1}{n} = \frac{1}{n}v_1$, $E=-\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2}=\frac{1}{n^2}E_1$.

1). $E=\frac12 mv^2-\frac{ke^2}{r}$

∵ $k\frac{e^2}{r^2}=m\frac{v^2}{r}$, ∴ $\frac12 mv^2=\frac{ke^2}{2r}$, $v=\sqrt{\frac{ke^2}{mr}}$.

∴ $E=-\frac{ke^2}{2r}$

将 $v=\sqrt{\frac{ke^2}{mr}}$ 代入 $mvr=n\frac{h}{2\pi}$ 得 $m^2r^2\frac{ke^2}{mr}=n^2\frac{h^2}{4\pi^2}$

解得 $r=\frac{\varepsilon_0 h^2}{\pi m e^2}n^2$. $E=-\frac{me^4}{8\varepsilon_0^2h^2}\frac{1}{n^2}$ ($E_1=-13.6eV$).

$v=\frac{e^2}{2\varepsilon_0 h}\frac1n$. ∴ $h\nu=\frac{me^4}{8\varepsilon_0^2h^2}(\frac{1}{j^2}-\frac{1}{i^2})$, 逸出功 $i\to\infty$, $W=\frac{me^4}{8\varepsilon_0^2h^2j^2}$

2). ∵ $pr=n\hbar$

∴ $E=\frac{p^2}{2m}-\frac{ke^2}{r}=\frac{p^2}{2m}-\frac{ke^2p}{n\hbar}=\frac{1}{2m}(p-\frac{kme^2}{n\hbar})^2-\frac{k^2me^4}{2\hbar^2}\cdot\frac{1}{n^2}$

∵能量最小时最稳定, ∴ $E_{min}=-\frac{k^2me^4}{2\hbar^2}\frac{1}{n^2}=-\frac{4\pi^2\cdot me^4}{16\pi^2\varepsilon_0^2\times 2h^2}\cdot\frac{1}{n^2}$

$=-\frac{me^4}{8\varepsilon_0^2h^2}\cdot\frac{1}{n^2}$.

3). 由不确定关系：$\Delta p\cdot\Delta r=\hbar$, $pr=n\hbar$.

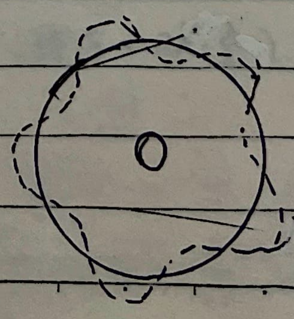
∴ $E=\frac{p^2}{2m}-\frac{ke^2}{r}$

$=\frac{1}{2m}\frac{n^2\hbar^2}{r^2}-\frac{ke^2}{r}=\frac{n^2\hbar^2}{2m}(\frac{1}{r^2}-\frac{2mke^2}{n^2\hbar^2}\frac1r+\frac{m^2k^2e^4}{n^4\hbar^4})-\frac{mk^2e^4}{2n^2\hbar^2}$.

$\geq -\frac{k^2me^4}{2\hbar^2}\frac{1}{n^2}$.

4) 驻波(物质波)条件：$2\pi r=n\lambda$.

∴ $r=\frac{n\lambda}{2\pi}=\frac{nh}{2\pi p}$ ∴ $pr=\frac{nh}{2\pi}=n\hbar$.



(物质波：德布罗意波 $E=mc^2=h\nu$

$p=\frac{h\nu}{c}=\frac{h}{\lambda}=mv$).

2. 光电效应.

$h\nu = \frac{1}{2}mv^2 + W$

$\begin{cases} E = h\nu \text{：光子能量.} \\ \frac{1}{2}mv^2\text{：光电子最大初动能.} \\ W\text{：逸出功.} \end{cases}$

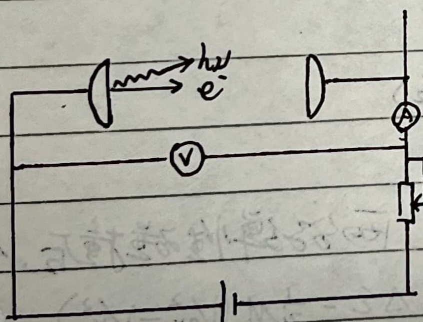
①截止频率(红限)：当 $v=0$ 时，$\nu_0 = \frac{W}{h}$. 对于特定金属，W 定.

②饱和电流：∵入射光强 $\Phi = N h\nu$，而光电流强度 $I = nev\Delta S$

$= \frac{N}{\Delta S\, v\Delta t} \cdot e\Delta S v = \frac{Ne}{\Delta t}$，∴ $\frac{\Phi}{\Delta t} = \frac{h\nu}{e} \cdot I$.

③反向截止电压：当 $eU_0 = \frac{1}{2}mv^2$ 时，光电流减小为零，∴ $U_0 = \frac{h\nu - W}{e}$

3. 逆光电效应.



$E = E' + h\nu$

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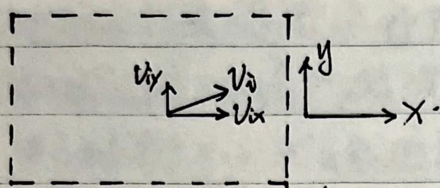
六. 理想气体

1. 压强:



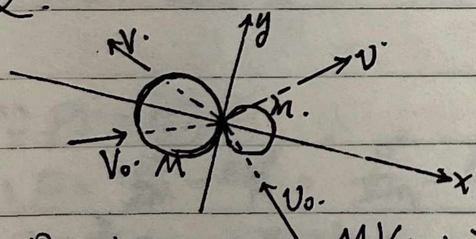
$$P=\frac{\Delta F}{\Delta S}=\frac{\sum_i I_i}{\Delta S\,\Delta t}=\frac{\frac{1}{2}\sum_i (m\cdot n_i v_{ix}\Delta t\cdot \Delta S)\cdot 2v_{ix}}{\Delta S\cdot \Delta t}$$

上式中 $m(n_i v_{ix}\Delta t\Delta S)$ 为 Δt 时间内一个长方体中撞击容器壁的分子的质量，$\frac{1}{2}$ 表示 v_{ix} 在 $\pm x$ 方向上概率都是 $\frac{1}{2}$，由于现在只研究分子撞击右壁的情况，故只取 $\frac{1}{2}$。∴ $P=nm\frac{\sum_i v_{ix}^2 n_i}{n}$

∵ $\overline{v_x^2}=\frac{\sum_i v_{ix}^2 n_i}{n}=\frac{1}{3}\overline{v^2}$

∴ $P=\frac{1}{3}nm\overline{v^2}=\frac{2}{3}n(\frac{1}{2}m\overline{v^2})$

2. 温度:



两分子弹性碰撞后，分子M损失动能:

$\Delta\varepsilon=\frac{1}{2}M(V_{0x}^2-V_x^2)$　　$(V_{0y}=V_y)$

$=\frac{1}{2}M(V_{0x}-V_x)(V_{0x}+V_x)$

由动量守恒: $V_x=\frac{MV_{0x}+mv_{0x}}{M+m}-\frac{m(V_{0x}-v_{0x})}{M+m}=\frac{(M-m)V_{0x}+2mv_{0x}}{M+m}$

∴ $V_{0x}-V_x=\frac{2m(V_{0x}-v_{0x})}{M+m}$，$V_{0x}+V_x=\frac{2(MV_{0x}+mv_{0x})}{M+m}$

∴ $\Delta\varepsilon=\frac{2mM}{(M+m)^2}[MV_{0x}^2-mv_{0x}^2-(M-m)v_{0x}V_{0x}]$

对大量M取平均，有 $\overline{\Delta\varepsilon}=\frac{2mM}{(M+m)^2}[M\overline{V_{0x}^2}-m\overline{v_{0x}^2}-(M-m)\overline{v_{0x}V_{0x}}]$.

∵ $\overline{V_{0x}}=\overline{v_{0x}}=0$，∴ $\overline{v_{0x}V_{0x}}=0$

∴ $\overline{\Delta\varepsilon}=\frac{2mM}{3(M+m)^2}(M\overline{V_0^2}-m\overline{v_0^2})$.　　$(\overline{v_x^2}=\frac{1}{3}\overline{v^2})$

$\propto \frac{1}{2}M\overline{V_0^2}-\frac{1}{2}m\overline{v_0^2}$

由热力学第零定律，分子平均动能在宏观上有温度特性，故设

$\frac{1}{2}m\overline{v^2}=B\cdot T$

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3. 理想气体状态方程：

由 $P=\frac{2}{3}n(\frac{1}{2}m\bar{v}^2)$ 和 $BT=\frac{1}{2}m\bar{v}^2$，有

$P=\frac{2}{3}nBT=\frac{2}{3}\frac{N}{V}BT=\frac{2}{3}\frac{\nu N_A}{V}BT$ （n：数密度，ν：摩尔数）

$\therefore PV=\frac{2}{3}\nu N_A BT$. 令 $R=\frac{2}{3}N_A B$，得

$PV=\nu RT$. $\because T=\frac{1}{B}(\frac{1}{2}m\bar{v}^2)=\frac{2N_A}{3R}(\frac{1}{2}m\bar{v}^2)$.

推论：$PV=\frac{M}{\mu}RT=\frac{\rho V}{\mu}RT$，$\therefore \mu P=\rho RT$

对于同一气体：$\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}$，$\frac{P_1}{\rho_1T_1}=\frac{P_2}{\rho_2T_2}$

对于同一系统的所有气体：$\frac{P_总V_总}{T_总}=R\sum_i n_i=\sum_i\frac{P_iV_i}{T_i}$

4. 热力学第一定律应用于理想气体 （n：摩尔数）

① 热力学过程：热力学系统状态随时间变化的过程（参量：P、T、V）.

- 准静态：每一中间态都近似为平衡态.
- 非静态：中间态不能视为平衡态（不能用 P-V 图表示）.

1). 功：$dW=Fdl=PSdl=PdV$

2) 热量：$dQ=nCdT$ （C：摩尔热容，对于固液C不变，对气体C可能变）

3) 内能：$dU=\frac{i}{2}nRdT$ （分子热运动：平动、转动、振动 + 分子内原子间键能）

$i=t+r+2s$ （t：平动自由度 $=3、2、1、0$；r：转动自由度 $=3N-6$）.

对于单原子分子，$U=N\cdot(\frac{1}{2}m\bar{v}^2)=N\cdot\frac{3R}{2N_A}T=\frac{3}{2}nRT$

热力学第一定律：$U_0+Q-W=U_t$，$dQ=dU+dW$

（吸热 Q 为正，对外做功 W 为正）.

② 各过程的求解 $\begin{cases} dQ=PdV+\frac{i}{2}nRdT \\ PV=nRT. \end{cases}$

1) 等容过程：$P=\frac{nR}{V}T$

- 系统对外界做功：$W=\int pdV=0$.
- 系统内能变化：$\Delta U=\frac{i}{2}nR\Delta T$.
- 系统从外界吸热：$Q=\frac{i}{2}nR\Delta T=nC_V\Delta T$.

2). 等压过程：$V=\frac{nR}{p}\cdot T$.

- 系统对外界做功：$W=\int p\,dV=p\Delta V=nR\Delta T$.
- 系统内能变化：$\Delta U=\frac{i}{2}nR\Delta T=nC_v\Delta T$.
- 系统从外界吸热：$Q=n(C_v+R)\Delta T=nC_p\Delta T$.

3) 等温过程：$p=nRT\cdot\frac{1}{V}$.

- 系统对外界做功：$W=\int_{V_1}^{V_2}p\,dV=nRT\int_{V_1}^{V_2}\frac{dV}{V}=nRT\ln\frac{V_2}{V_1}=-nRT\ln\frac{P_2}{P_1}$.
- 系统内能变化：$\Delta U=\frac{i}{2}nR\Delta T=0$.
- 系统从外界吸热：$Q=nRT\ln\frac{V_2}{V_1}$.

4) 绝热过程：$PV^{\frac{C_p}{C_v}}=P_0V_0^{\frac{C_p}{C_v}}$.

由 $PdV+Vdp=nRdT$ 和 $nC_vdT+PdV=dU+dW=0$ 得

$\frac{C_v}{R}(RdV+Vdp)+PdV=0$，即 $C_p\frac{dV}{V}=-C_v\frac{dp}{p}$.

积分得 $\ln\frac{V}{V_0}=-\frac{C_v}{C_p}\ln\frac{P}{P_0}=\ln\left(\frac{P_0}{P}\right)^{\frac{C_v}{C_p}}$，$\therefore PV^{\frac{C_p}{C_v}}=P_0V_0^{\frac{C_p}{C_v}}$.

- 系统对外界做功：$W=\int_{V_1}^{V_2}PdV=\int_{V_1}^{V_2}\frac{P_0V_0^{\frac{C_p}{C_v}}}{V^{\frac{C_p}{C_v}}}dV$

$=P_0V_0^{\frac{C_p}{C_v}}\int_{V_1}^{V_2}V^{-\frac{C_p}{C_v}}dV=P_0V_0^{\frac{C_p}{C_v}}\cdot\frac{1}{1-\frac{C_p}{C_v}}\cdot V^{1-\frac{C_p}{C_v}}\Big|_{V_1}^{V_2}=-\frac{C_v}{R}P_0V_0^{\frac{C_p}{C_v}}\left(V_2^{-\frac{R}{C_v}}-V_1^{-\frac{R}{C_v}}\right)$

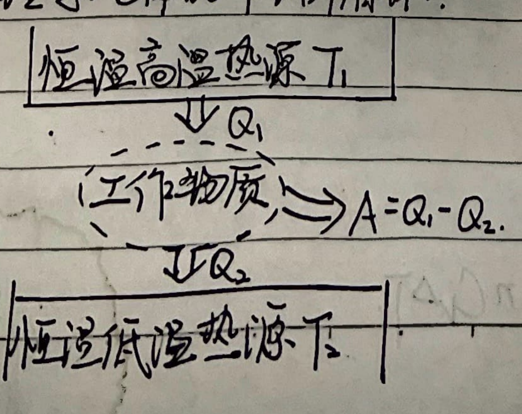
$$=\begin{cases}\frac{C_v}{R}P_1V_1^{\frac{C_p}{C_v}}\left(V_1^{-\frac{R}{C_v}}-V_2^{-\frac{R}{C_v}}\right)=\frac{C_v}{R}P_1V_1\left[1-\left(\frac{V_1}{V_2}\right)^{\frac{R}{C_v}}\right]\\ \frac{C_v}{R}P_2V_2^{\frac{C_p}{C_v}}\left(V_1^{-\frac{R}{C_v}}-V_2^{-\frac{R}{C_v}}\right)=\frac{C_v}{R}P_2V_2\left[\left(\frac{V_2}{V_1}\right)^{\frac{R}{C_v}}-1\right]\end{cases}$$

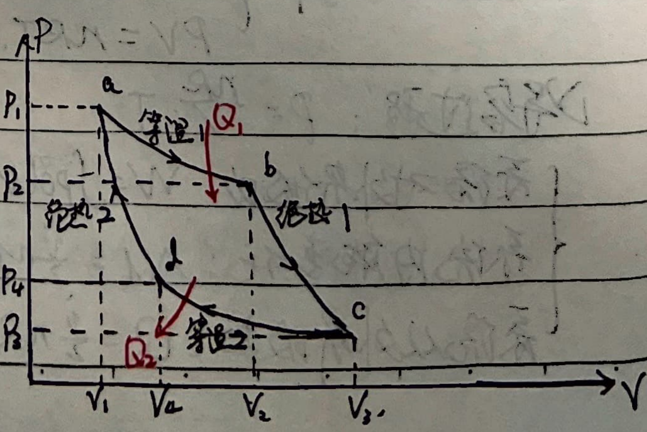
$\therefore P_1V_1\left[1-\left(\frac{V_1}{V_2}\right)^{\frac{R}{C_v}}\right]=P_2V_2\left[\left(\frac{V_2}{V_1}\right)^{\frac{R}{C_v}}-1\right]$ 解得 $\left(\frac{V_1}{V_2}\right)^{\frac{R}{C_v}}=\frac{P_2V_2}{P_1V_1}$

$\therefore W=\frac{C_v}{R}P_1V_1\left(1-\frac{P_2V_2}{P_1V_1}\right)=\frac{C_v}{R}(P_1V_1-P_2V_2)=nC_v(T_1-T_2)$.

- 系统内能变化：$\Delta U=nC_v(T_2-T_1)=nC_v\Delta T$.
- 系统从外界吸热：$Q=0$.

5) 理想气体的卡诺循环.





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(1) $Q_1 = nRT_1 \ln\frac{V_2}{V_1}$：等温膨胀 ab 中从高温热源吸热 Q_1.

(2) $T_1V_2^{\frac{Cp}{Cv}-1} = T_2V_3^{\frac{Cp}{Cv}-1}$：绝热过程 bc

(3) $Q_2 = nRT_2 \ln\frac{V_3}{V_4}$：等温压缩 cd 中向低温热源放热 Q_2.

(4) $T_2V_4^{\frac{Cp}{Cv}-1} = T_1V_1^{\frac{Cp}{Cv}-1}$：绝热过程 da.

由(2)(4)得：$\frac{V_2}{V_1} = \frac{V_3}{V_4}$，代入(3)，得

$Q_2 = nRT_2 \ln\frac{V_2}{V_1}$，与(1)比较，得

$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

$\therefore \eta = \frac{A}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$.

5. 饱和汽和未饱和汽.（$n_{蒸发} = n_{返回}$ / $n_{蒸发} > n_{返回}$）

性质：① 同温度下，不同液体饱和汽压一般不同；

② 一定液体饱和汽压随温度升高而迅速升高.

故当饱和汽体积减小或温度降低时，汽将液化.

③ 液体沸腾的条件是液体在该温度下的饱和汽压等于外界压强.

计算：根据道尔顿分压定律：密闭容器中的气压 $P = P_{空气} + P_{蒸汽}$，

当容器内仍有液态水并处于平衡态时，$P_{蒸汽} = P_{饱}$，即相对湿度为100%.

对于空气和水蒸汽共存的混合气体，当蒸汽未饱和时，近似遵循

理想气体状态方程，即 $P_{空气}V = n_{空气}RT$，$P_{蒸汽}V = n_{蒸汽}RT$，

及 $PV = n_{总}RT$，当温度降低时，水蒸汽变成饱和水蒸汽，再降温或

加压时，水蒸汽将部分液化，此时：$\frac{P_{空气}V}{T} = \frac{P'_{空气}V'}{T'}$，而

$\frac{P_{饱}V}{TR} = \frac{P'_{饱}V'}{T'R} + n_{液化}$.

对于空气和水蒸汽共存，并没有液态水存在时，可用理想气体状态方程计算.

$\rho = \frac{\mu_{空气} \cdot n_{空气} + \mu_{蒸汽} \cdot n_{蒸汽}}{V}$；（$\mu_{空气} = 28.8g/mol$，$\mu_{蒸汽} = 18g/mol$）.

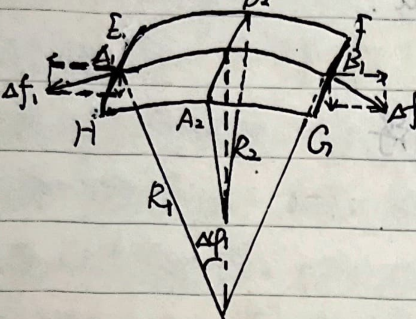
并定义：绝对湿度 $= P_{蒸汽}$，相对湿度 $= \frac{P_{蒸汽}}{P_{饱}}$（$P_{蒸汽}$：该时刻蒸汽压；$P_{饱}$：该温度下饱和汽压）

七. 液体与固体

1. 液体的表面张力

$F = \sigma L$　方向: 沿所划线法向，液面切向.

性质: 对于非平面液体表面，表面张力会引起内外压强差. 如图:



设 $\overset{\frown}{A_1B_1}=\overset{\frown}{EF}=\overset{\frown}{HG}=\Delta l_2$

$\overset{\frown}{A_2B_2}=\overset{\frown}{HE}=\overset{\frown}{GF}=\Delta l_1$.

由于 Δf_1 和 Δf_2 的水平分量互相抵消，

∴ Δf_1 与 Δf_2 的合作用为 $(\Delta f_1+\Delta f_2)\Delta\varphi$

$= 2\sigma\Delta l_1\Delta\varphi = 2\sigma\Delta l_1\cdot\frac{A_1B_1}{2R_1} = \frac{\sigma}{R_1}\Delta l_1\Delta l_2 = \frac{\sigma}{R_1}\Delta S$

同理对于 $\overset{\frown}{A_2B_2}$ 也有合作用 $\frac{\sigma}{R_2}\Delta S$.

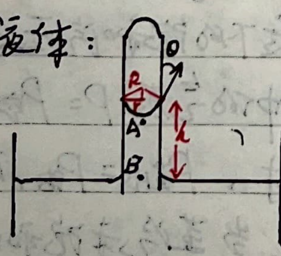
∴ $\Delta f_{总} = \sigma(\frac{1}{R_1}+\frac{1}{R_2})\Delta S$，∴ 弯曲液面引起的内外压强差 $p = \frac{\Delta f_{总}}{\Delta S} = \sigma(\frac{1}{R_1}+\frac{1}{R_2})$

其中 R_1 R_2 为一对相垂直的正截口曲线的曲率半径.

对球面，附加压强 $p_1 = \frac{2\sigma}{R}$
对球膜，附加压强 $p_2 = 2p_1 = \frac{4\sigma}{R}$.

① 毛细现象

对于浸润液体:

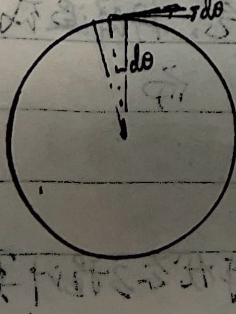


∵ $p_A = p_0 - \frac{2\sigma}{R}$.

$p_B = p_A + \rho g h = p_0$.

∴ $h = \frac{2\sigma}{\rho g R}$. 又∵ $R\cos\theta = r$，∴ $h = \frac{2\sigma\cos\theta}{\rho g r}$

② 肥皂泡的振动.



当肥皂泡径向膨胀 x 时，面元上有.

$dF_{合} = p_0 dS + \frac{4\sigma}{R+x}dS - p'dS$.　($p_0$ 为大气压).

平衡时，有 $p_0 dS + \frac{4\sigma}{R}dS = p\,dS$.　∴ $p = p_0 + \frac{4\sigma}{R}$.

由于等温变化，∴ $p\cdot\frac{4}{3}\pi R^3 = p'\frac{4}{3}\pi(R+x)^3$　∴ $p' = (\frac{R}{R+x})^3\cdot(p_0+\frac{4\sigma}{R})$.

∴ $dF_{合} = p_0[1-(\frac{R}{R+x})^3]dS + 4\sigma[\frac{1}{R+x} - \frac{R^2}{(R+x)^3}]dS$

$\approx p_0[1-\frac{R^3}{R^3+3R^2x}]dS + 4\sigma[\frac{1}{R+x} - \frac{R^2}{R^3+3R^2x}]dS$

$\approx p_0(1-\frac{R-3x}{R})dS + 4\sigma(\frac{R-x}{R^2} - \frac{R-3x}{R^2})dS$

$= \frac{3x}{R} p_0 ds + \frac{2x}{R} \cdot \frac{4\sigma}{R} ds.$

$\therefore dk = \frac{3p_0 ds}{R} + \frac{8\sigma ds}{R^2}.$

$\therefore T = 2\pi\sqrt{\frac{dm}{dk}} = 2\pi\sqrt{\frac{\frac{ds}{4\pi R^2} m}{(\frac{3p_0}{R} + \frac{8\sigma}{R^2}) ds}} = 2\pi\sqrt{\frac{m}{12\pi R p_0 + 32\pi\sigma}}.$

此过程中表面张力做功 $W = \int_{R_1}^{R_2} \frac{4\sigma}{R} \cdot 4\pi R^2 \cdot dR = 16\pi\sigma\int_{R_1}^{R_2} R dR = 8\pi\sigma(R_2^2 - R_1^2).$

同样，用表面自由能计算：$W = (2\sigma)(S_2 - S_1) = 2\sigma(4\pi R_2^2 - 4\pi R_1^2) = 8\pi\sigma(R_2^2 - R_1^2)$

(因为液面有两层，每层做功 $\sigma\Delta S$，两层克服表面张力做功共 $2\sigma\Delta S$)。

2. 固体：

① 单晶体：各向异性.

多晶体：各向同性.

非晶体：各向同性.

液体：各向同性

液晶：各向异性.

② 热膨胀：设固体在0°C和t°C时长度分别为l_0和l. 有

$l = l_0(1+\alpha t)$, $dl = \alpha l_0 dt$.

设固体在0°C和t°C时体积分别为V_0和V. 有

$V = V_0(1+\beta t)$, $dV = \beta V_0 dt$

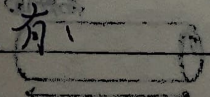
且 $\beta = 3\alpha$.

③ 对于同一固体，熔解热 $Q_1 = \lambda m$. 汽化热 $Q_2 = Lm$.

升华热 $Q_3 = Q_1 + Q_2 = (\lambda + L)m$.

④ 热传导：设一块横截面积为S、厚度为x的板，板两边保持不同温度ΔT.

在dt时间内垂直于板两面流过的热量为dQ. 有：



$H = \frac{dQ}{dt} = -kS \cdot \frac{\Delta T}{x}$

$\begin{cases} \text{多层板串联：} H = \frac{-S\Delta T}{\sum(x_i/k_i)} \\ \text{多层板并联：} H = -\frac{\Delta T}{x}\sum(k_i S_i) \end{cases}$ (类比电阻、电容、电感、弹簧的串并联).

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八. 静电场

1. 电场力：$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\cdot\hat{r}$.

2. 场强：$\vec{E} = \frac{\vec{F}}{q_0}$ $\begin{cases} \vec{E}_合 = \sum E_i \text{（对于离散点电荷系统）} \\ \vec{E}_合 = \int d\vec{E} \text{（对于连续带电体）} \end{cases}$

在静电场中 $\begin{cases} \text{高斯定理：} \Phi_e = \oint_S \vec{E}\, d\vec{S} = \frac{q}{\varepsilon_0} \text{（q为高斯面包围的净电荷）} \\ \text{环路定理：} \oint_L \vec{E}\, d\vec{l} = 0. \end{cases}$

① 点电荷场强分布：

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}.$$

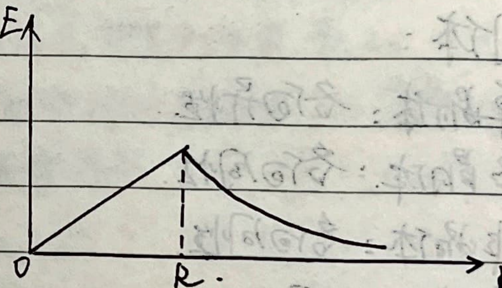
② 实心球（均匀带电实心球，电荷体密度处处相等，电荷不能自由移动）场强分布

$r<R$ 时，$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{r^3}{R^3}Q$

$\therefore E = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}$

$r \geq R$ 时，$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}Q$

$\therefore E = \frac{Q}{4\pi\varepsilon_0 r^2}$



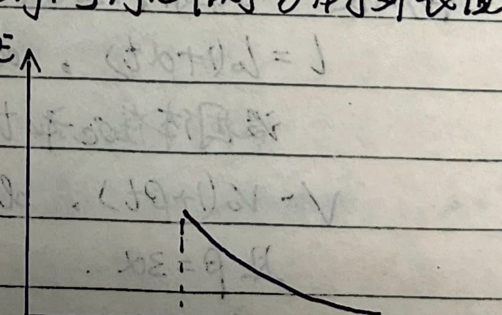
③ 空球壳（薄导体球壳或实心导体球，电荷由于静电平衡分布于外表面，厚度线度，电荷可自由移动）场强分布

$r<R$ 时，$E\cdot 4\pi r^2 = 0$

$\therefore E = 0$

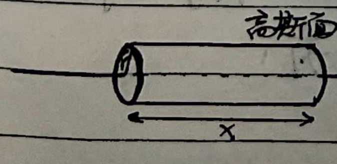
$r \geq R$ 时，$E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}Q$

$\therefore E = \frac{Q}{4\pi\varepsilon_0 r^2}$



④ 无限长导线（电荷线密度λ）场强分布

$E\cdot 2\pi r\cdot x = \frac{1}{\varepsilon_0}\lambda\cdot x$



$\therefore E = \frac{\lambda}{2\pi\varepsilon_0 r}$

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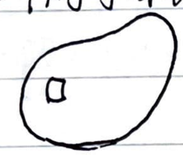
⑤无限宽平板（电荷面密度σ）场强分布.

$$2E\cdot S=\frac{1}{\varepsilon_0}S\cdot\sigma$$

$$\therefore E=\frac{\sigma}{2\varepsilon_0}$$

⑥静电平衡导体表面

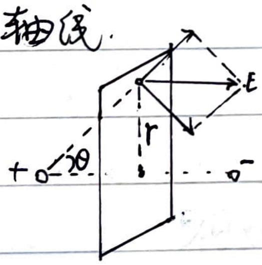


$$E\cdot\Delta S=\frac{1}{\varepsilon_0}\Delta S\cdot\sigma$$

$$\therefore E=\frac{\sigma}{\varepsilon_0}.$$

⑦电偶极子.（电矩 $\vec{P}=q\vec{l}$，$\vec{l}$ 由 $-q$ 指向 $+q$）.

1)轴线.

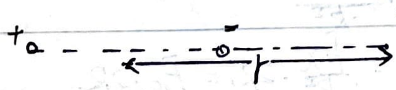


$$E=2E^+\cos\theta=2\frac{kq}{r^2+(\frac{l}{2})^2}\cdot\frac{\frac{l}{2}}{\sqrt{r^2+(\frac{l}{2})^2}}=\frac{kql}{(r^2+\frac{l^2}{4})^{\frac{3}{2}}}$$

当 $r\gg l$ 时，有 $r_+=r_-=r$.

$$\therefore E\approx\frac{kql}{r^3}=\frac{kp}{r^3}.$$

2)连线

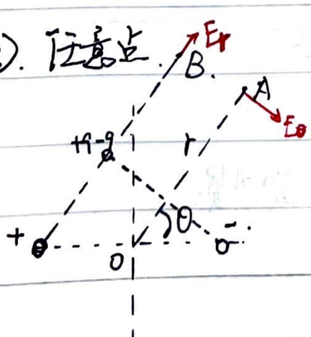


$$E=\frac{kq}{(r-\frac{l}{2})^2}-\frac{kq}{(r+\frac{l}{2})^2}$$

$$=kq\frac{r^2+rl+\frac{l^2}{4}-r^2+rl-\frac{l^2}{4}}{[r^2-(\frac{l}{2})^2]^2}\approx kq\cdot\frac{2rl}{r^4}=\frac{2kql}{r^3}.$$

$$\therefore E\approx\frac{2kp}{r^3}$$

3).任意点

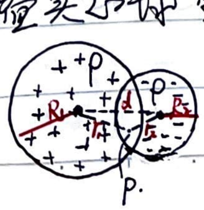


以OA为对称轴作出 $-q$ 的对称电荷 $+q$，在同处再加一 $-q$.

于是组成两个电偶极子. $\therefore E_\theta\approx\frac{kql\sin\theta}{r^3}$，$E_r\approx\frac{2kql\cos\theta}{r^3}$

$$\therefore E=\sqrt{E_\theta^2+E_r^2}=\frac{kql}{r^3}\sqrt{\sin^2\theta+4\cos^2\theta}=\frac{kql}{r^3}\sqrt{1+3\cos^2\theta}.$$

⑧.重叠实心球重合部分场强（体密度ρ）.



在重合部分剩在 $+\rho$ 和 $-\rho$ 的电荷，有 $\vec{E_P}=\frac{k\cdot\frac{4}{3}\pi r_1^3\rho}{r_1^3}\vec{r_1}-\frac{k\frac{4}{3}\pi r_2^3\rho}{r_2^3}\vec{r_2}$

$$=k\frac{4}{3}\pi\rho(\vec{r_1}-\vec{r_2})=\frac{\rho\vec{d}}{3\varepsilon_0}.$$

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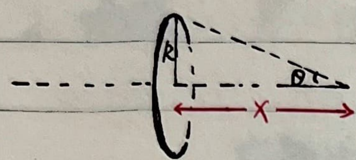
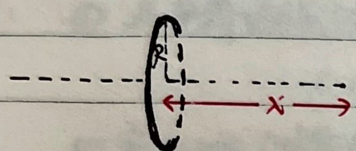
⑨ 带电圆环轴线上场强分布.



$$E=\int\frac{kdq}{x^2+R^2}\cos\theta=\frac{kq}{x^2+R^2}\cdot\frac{x}{\sqrt{x^2+R^2}}$$
$$=\frac{qx}{4\pi\varepsilon_0(x^2+R^2)^{\frac{3}{2}}}.$$

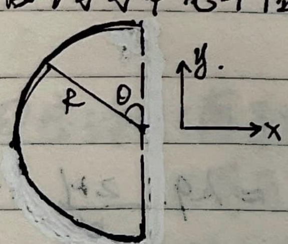
⑩ 带电圆板轴线上场强分布:



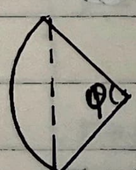
利用上面结论 $E=\int\frac{dq\,x}{4\pi\varepsilon_0(x^2+R^2)^{\frac{3}{2}}}$
$$=\int_0^R\frac{(2\pi r\,dr\,\sigma)x}{4\pi\varepsilon_0(x^2+r^2)^{\frac{3}{2}}}=\frac{\sigma x}{2\varepsilon_0}\int_0^R\frac{r\,dr}{(x^2+r^2)^{\frac{3}{2}}}$$
$$=\frac{\sigma x}{4\varepsilon_0}\int\frac{d(x^2+r^2)}{(x^2+r^2)^{\frac{3}{2}}}=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right).$$

专题: 场强等效求法.

1. 由均匀带电半圆环圆心处场强推广为张角为 φ 的弧.

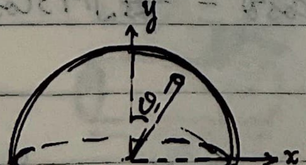


$$\begin{cases}dE_y=0\ (\text{对称性})\\ dE_x=\frac{k(R\,d\theta)\lambda}{R^2}\sin\theta=\frac{k\lambda\,dy}{R^2}\end{cases}$$
$$\therefore E=\int dE_x=\frac{k\lambda}{R^2}\int_{-R}^{R}dy=\frac{k\lambda}{R^2}\cdot 2R=E_0\cdot 2R.$$

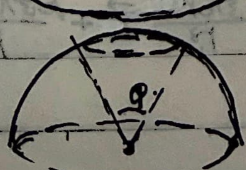


∴对张角为 φ 的弧
$$E=\frac{k\lambda}{R^2}\int dy=\frac{k\lambda}{R^2}\cdot\left(2R\sin\frac{\varphi}{2}\right)$$

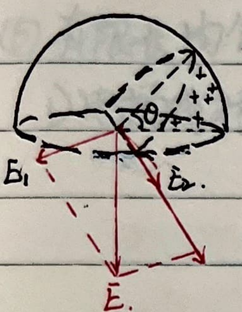
2. 由均匀带电半球球心处场强推广为张角为 φ 的球冠.



$$\begin{cases}dE_x=0\ (\text{对称性})\\ dE_y=\frac{k\sigma\,ds\cos\theta}{R^2}=\frac{k\sigma\,ds_x}{R^2}\end{cases}$$
$$\therefore E=\int dE_y=\frac{k\sigma}{R^2}\int ds_x=\frac{k\sigma}{R^2}\pi R^2=E_0\pi R^2.$$



∴对张角为 φ 的球冠
$$E=\frac{k\sigma}{R^2}\int ds_x=\frac{k\sigma}{R^2}\cdot\pi\left(R\sin\frac{\varphi}{2}\right)^2.$$

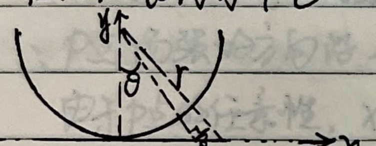


$$\because \begin{cases} E_1\cos\frac{\theta}{2} = E_2\cos\frac{\pi-\theta}{2} \\ E_1\sin\frac{\theta}{2} + E_2\sin\frac{\pi-\theta}{2} = E_0\cdot\pi R^2 \end{cases}$$

解得 $E = (E_0\cdot\pi R^2)\sin\frac{\theta}{2}$

$= \frac{kq}{R^2}(\pi R^2)\cdot\sin\frac{\theta}{2}$.

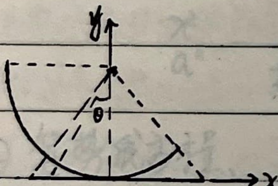
3. 由无限长均匀带电直线外R处场强推广为有限长导线.



$$\begin{cases} dE_x = 0 \text{（对称性）} \\ dE_y = \frac{k\lambda dl}{r^2}\cos\theta = \frac{k\lambda\frac{\left(\frac{R}{\cos\theta}\right)d\theta}{\cos\theta}}{\left(\frac{R}{\cos\theta}\right)^2}\cdot\cos\theta = \frac{k\lambda}{R}\cos\theta d\theta \end{cases}$$

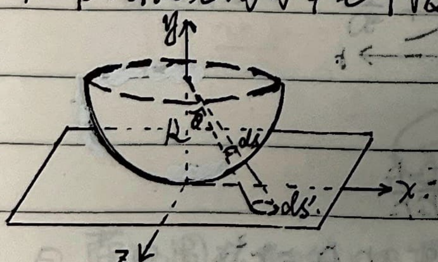
$$= \frac{k(\lambda R d\theta)}{R^2}\cos\theta.$$

等价于一个半圆导线在其圆心处的场强.



$$\begin{cases} dE_x = \frac{k\lambda dl}{r^2}\sin\theta = \frac{k(\lambda R d\theta)}{R^2}\sin\theta. \\ dE_y = \frac{k\lambda dl}{r^2}\cos\theta = \frac{k(\lambda R d\theta)}{R^2}\cos\theta. \end{cases}$$

4. 由无限宽均匀带电平板外R处场强推广为长宽有限的平板：

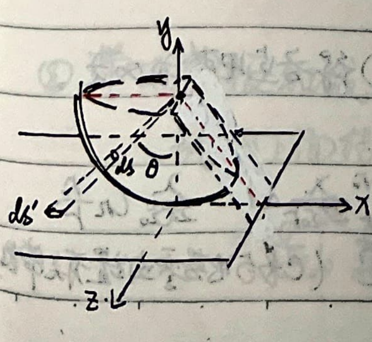


$$\begin{cases} dE_x = dE_z = 0 \text{（对称性）}. \\ dE_y = \frac{k\sigma ds'}{r^2}\cos\theta \end{cases}$$

$\because \frac{ds'\cos\theta}{\left(\frac{R}{\cos\theta}\right)^2} = \frac{ds}{R^2}$ $\quad\therefore ds' = \frac{ds}{\cos^3\theta}$.

$\therefore dE_y = \frac{k\sigma\cdot\frac{ds}{\cos^3\theta}}{\left(\frac{R}{\cos\theta}\right)^2}\cdot\cos\theta = \frac{k\sigma ds}{R^2}$.（与直导线不同）.

$\therefore E = \int dE_y = \frac{k\sigma}{R^2}\cdot 2\pi R^2 = 2\pi k\sigma = \frac{\sigma}{2\varepsilon_0}$.



$$\begin{cases} dE_x = \frac{k\sigma ds'}{r^2}\sin\theta\cos\alpha = \frac{k\sigma ds}{R^2}\tan\theta\cos\alpha \\ dE_y = \frac{k\sigma ds'}{r^2}\cos\theta = \frac{k\sigma ds}{R^2}. \\ dE_z = 0 \end{cases}$$

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3. 电势：$\varphi_a=\int_a^\infty \vec{E}\cdot d\vec{l}$ $\begin{cases} \varphi_a=\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} & 离散. \\ \varphi_a=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r} & 连续. \end{cases}$

当无限远处无导体时，无限远处电势为零.

不考虑地球所带电荷时，大地视为电势为零.

① 点电荷电势分布.

$$\varphi_a=\frac{q}{4\pi\varepsilon_0}\int_a^\infty\frac{dr}{r^2}=\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Big|_a^\infty=\frac{q}{4\pi\varepsilon_0 a}.$$

$$U_{ab}=\varphi_a-\varphi_b=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

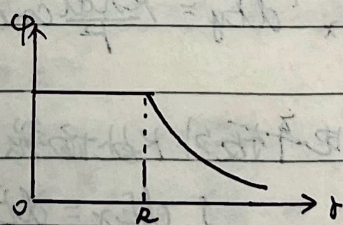
② 实心球

$$\begin{cases} r\geq R时，\varphi=\int_r^\infty \vec{E}\cdot d\vec{l}=\frac{q}{4\pi\varepsilon_0 r}. \\ r<R时，\varphi=\frac{q}{4\pi\varepsilon_0 R}+\int_r^R\frac{q}{4\pi\varepsilon_0}\cdot\frac{r\,dr}{R^3} \end{cases}$$

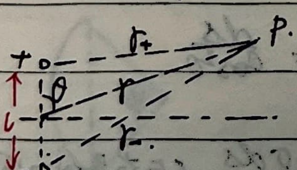
$$=\frac{q}{4\pi\varepsilon_0 R}+\frac{q(R^2-r^2)}{8\pi\varepsilon_0 R^3}=\frac{q(3R^2-r^2)}{8\pi\varepsilon_0 R^3}.$$

③ 空球壳.

$$\begin{cases} r\geq R时，\varphi=\frac{q}{4\pi\varepsilon_0 r} \\ r<R时，\varphi=\frac{q}{4\pi\varepsilon_0 R} \end{cases}$$



④ 电偶极子.



$$\varphi=\frac{q}{4\pi\varepsilon_0 r_+}-\frac{q}{4\pi\varepsilon_0 r_-}=\frac{q}{4\pi\varepsilon_0}\frac{r_--r_+}{r_+r_-}$$

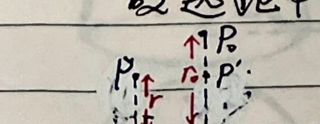
当 $r\gg l$ 时，有 $r_--r_+=l\cos\theta$，$r_-r_+=r^2$.

$$\therefore\varphi\approx\frac{q}{4\pi\varepsilon_0}\frac{l\cos\theta}{r^2}=\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^2}.$$

⑤ 无限长均匀带电直线电势分布.

若仍选无限远为电势零点，则由 $\int_P^\infty \vec{E}\cdot d\vec{r}$ $\left(E=\frac{\lambda}{2\pi\varepsilon_0 r}\right)$ 得各点电势为∞大.

故选距带电直线 r_0 的 P_0 点为电势零点，如图.



$$\varphi_P=\int_P^{P'}\vec{E}\cdot d\vec{r}+\int_{P'}^{P_0}\vec{E}\cdot d\vec{r}=\int_{P'}^{P_0}\vec{E}\cdot d\vec{r}=\int_r^{r_0}\frac{\lambda}{2\pi\varepsilon_0 r}dr=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

$$=-\frac{\lambda}{2\pi\varepsilon_0}\ln r+\frac{\lambda}{2\pi\varepsilon_0}\ln r_0=-\frac{\lambda}{2\pi\varepsilon_0}\ln r+C$$ （C为与电势零点位置有关常数）

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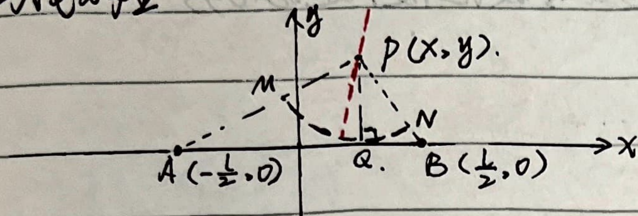
⑥ 有限长均匀带电直线电场线与等势线方程.

1) 电场线方程



由前面的推导, AB 对 P 的作用等效于 $\widehat{MN}$ (线密度同为λ) 对 P 的效果.

∴ P 点场强的方向沿 ∠MPN 角平分线方向, 也即电场线的切线方向.

由于 P 点的任意性, xy 平面上任意一点场强方向都沿 ∠APB 的平分线方向.

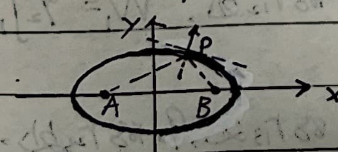
由双曲线的性质 (其上任意一点切线方向与由两焦点到该点连线的夹角平分线方向相同) 可知, 电场线为同焦点的双曲线族, 其方程为

$$\frac{x^2}{a^2}-\frac{y^2}{\frac{L^2}{4}-a^2}=1 \quad 其中\ 0<a<\frac{L}{2}.$$

2) 等势线方程,

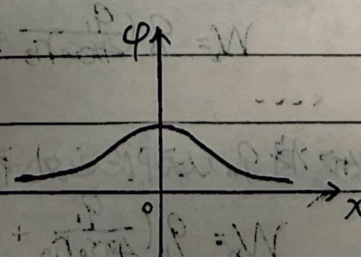
由于电场线方程是双曲线, 而等势线处处与电场线垂直, 故等势线方程为椭圆:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-\frac{L^2}{4}}=1. \quad 其中\ a>\frac{L}{2}.$$



⑦ 带电圆环轴线上电势分布.

$$\varphi=\int\frac{dq}{4\pi\varepsilon_0\sqrt{x^2+R^2}}=\frac{q}{4\pi\varepsilon_0\sqrt{x^2+R^2}}$$



⑧ 带电圆板轴线上电势分布.

利用上面结论 $\varphi=\int\frac{dq}{4\pi\varepsilon_0\sqrt{x^2+r^2}}=\int_0^R\frac{2\pi r\,dr\,\sigma}{4\pi\varepsilon_0\sqrt{x^2+r^2}}$

$$=\frac{\sigma}{2\varepsilon_0}\int_0^R\frac{r\,dr}{\sqrt{x^2+r^2}}=\frac{\sigma}{4\varepsilon_0}\int\frac{d(x^2+r^2)}{\sqrt{x^2+r^2}}=\frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+R^2}-x\right).$$

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∴ ⑤ 无限宽均匀带电平板电势分布

选带电平面本身为电势零点，故距板 r 处的电势

$\varphi = -\int_0^r \frac{\sigma}{2\varepsilon_0} dr = -\frac{\sigma r}{2\varepsilon_0}$

4. 电势能、静电能

① 电势能：由于静电场是保守场，在静电场中移动电荷时，静电场力做功与路径无关，故任一电荷在静电场中都有静电势能

$W = q\varphi$

$A_{12} = W_1 - W_2 = q(\varphi_1 - \varphi_2)$

一个电荷在外电场中的电势能是属于该电荷与产生电场的电荷系所共有的，是一种相互作用能；同理，点电荷系的电势能为

$W = \sum q_i U_i$，U_i 不包括此电荷系内电荷在该系中其它电荷处产生的电势

② 静电能：1) 互能：将各电荷从现位置彼此分散到无限远处，它们之间静电力所做的功，即点电荷系的相互作用能

〈1〉将 q_1 从无限远处移至空间某点，不做功。

〈2〉将 q_2 从无限远处移至 q_1 的 r_{12} 处，$W_1 = q_2\int_{r_{12}}^{\infty} \frac{q_1\, dr}{4\pi\varepsilon_0 r^2}$

$= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}$

〈3〉将 q_3 从无限远处移至 q_1 的 r_{13} 处，q_2 的 r_{23} 处。

$W_2 = q_3\left(\frac{q_1}{4\pi\varepsilon_0 r_{13}} + \frac{q_2}{4\pi\varepsilon_0 r_{23}}\right)$

……

〈n〉将 q_n 从无限远处移至 q_1 的 r_{1n} 处 …… q_{n-1} 的 $r_{(n-1)n}$ 处。

$W_n = q_n\left(\frac{q_1}{4\pi\varepsilon_0 r_{1n}} + \frac{q_2}{4\pi\varepsilon_0 r_{2n}} + \cdots + \frac{q_{n-1}}{4\pi\varepsilon_0 r_{(n-1)n}}\right)$

∴ $E = \sum_{i=1} W_i = \frac{1}{2}\sum_{i=1} q_i \varphi_i$

2) 自能：将一带电体分割成无限电荷元，并分散到彼此的无限远处，它们之间静电力所做的功。

$E = \frac{1}{2}\int_q \varphi\, dq$

故空间中的静电能 = 互能 + 自能 = $\frac{1}{2}\int \varphi\, dq$.

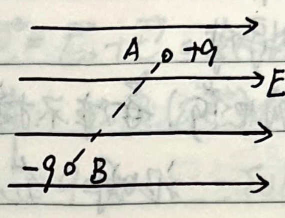
$= \int \omega\, dV = \frac{1}{2}\varepsilon_0 \int E^2 dV$ （ω：能量密度）.

⑤ (1). 电偶极子在外电场（匀强E）中的电势能

$W = \sum_{i=1}^{2} q_i U_i = q\varphi_A + (-q\varphi_B)$

$= -qEl\cos\theta$

$= -\vec{p}\cdot\vec{E}$



(2) 平行板电容器静电能：

$W = \frac{1}{2}\int \varphi\, dq = \frac{1}{2}q(U_1 - U_2) = \frac{1}{2}qU$.

$W = \frac{1}{2}\varepsilon_0 \int E^2 dV = \frac{1}{2}\varepsilon_0 E^2 S\cdot d = \frac{1}{2}\varepsilon_0 \frac{U^2}{d^2} S\cdot d = \frac{1}{2}CU^2$.

$W = \int q\, dU = \int \frac{q\, dq}{C} = \frac{1}{2}\frac{q^2}{C}$.

(3) 实心球

$r<R$ 时，$W = \frac{1}{2}\varepsilon_0 \int_0^R \left(\frac{q}{4\pi\varepsilon_0}\frac{r}{R^3}\right)^2 4\pi r^2 dr = \frac{1}{2}\varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)^2 4\pi \cdot \int_0^R r^4 dr = \frac{q^2}{40\pi\varepsilon_0 R}$;

$r \geq R$ 时，$W = \frac{1}{2}\varepsilon_0 \int_R^\infty \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr = \frac{1}{2}\varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \cdot 4\pi \int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0 R}$

$\therefore W_总 = \frac{q^2}{40\pi\varepsilon_0 R} + \frac{q^2}{8\pi\varepsilon_0 R} = \frac{3q^2}{20\pi\varepsilon_0 R}$;

$W_总 = \frac{1}{2}\int \varphi\, dq = \frac{1}{2}\int_0^R \frac{(3R^2 - r^2)q}{8\pi\varepsilon_0 R^3} \cdot \frac{q}{\frac{4}{3}\pi R^3} 4\pi r^2 dr = \frac{3q^2}{20\pi\varepsilon_0 R}$

(4) 空球壳

$r<R$ 时，$W=0$ ；$r \geq R$ 时，$W = \frac{1}{2}\varepsilon_0 \int_R^\infty \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\varepsilon_0 R}$.

$\therefore W_总 = \frac{q^2}{8\pi\varepsilon_0 R}$

$W_总 = \frac{1}{2}\int \varphi\, dq = \frac{1}{2}\cdot \frac{q}{4\pi\varepsilon_0 R}\cdot q = \frac{q^2}{8\pi\varepsilon_0 R}$.

5. 导体与电介质

① 静电平衡时的导体：

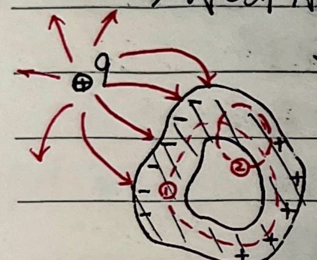
内：$\vec{E} = \vec{E_0}(外加) + \vec{E'}(感应) = 0$.

外：导体外部表面场强 $\vec{E_s} = \frac{\sigma}{\varepsilon_0}\vec{e}$ 垂直于外表面。

} 导体是等势体、表面是等势面，感应电荷分布于表面

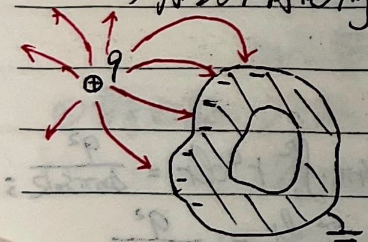
带空腔导体：

1) 腔外有电荷，导体不接地，内壁无电荷，腔内无电场.



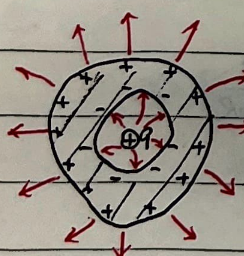
证明：在导体内部取高斯面①包围空腔，由于 $\Phi = \oint_S \vec{E}\cdot d\vec{S} = 0$，内壁净电荷为零；又取一环路②，根据环路定理，腔内无电场线，故内壁无电荷，腔内无电场.

2) 腔外有电荷，导体接地，内壁无电荷，外表面只有一种电荷，腔内无电场.



证明：若有正电荷存在于导体外表面（如图），则其发出的电场线将伸向无限远（不可能止于导体另一面的负电荷，因两处电势相等），而无限远处电势也为零，与导体与无限远处有电场线相连矛盾.

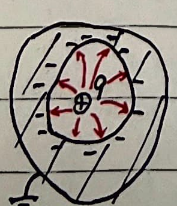
3) 腔内有电荷，导体不接地，内外壁电荷分别为 $-q$，$+q$，腔外有电场（如图）



证明：取一高斯面在导体内部包围空腔，由于 $\oint_S \vec{E}\cdot d\vec{S}=0$ 内壁电荷为 $-q$ 以抵消空腔内的电荷 $+q$.

又取一高斯面在外部包围导体，由于电荷守恒，外表面有 $+q$，故净电荷为 $+q$，$\oint_S \vec{E}\cdot d\vec{S} \neq 0$，腔外有电场.

4) 腔内有电荷，导体接地，内壁电荷 $-q$，外壁电荷为0，腔外无电场（如图）



证明：取一高斯面在导体内部包围空腔，由于 $\oint_S \vec{E}\cdot d\vec{S}=0$ 内壁有 $-q$ 的电荷；又由于导体内部，包括外表面电势为零，而无限远处电势也为零，所以腔外无电场. 综上，1) 2)情况中外不影响内、3)情况内不影响外，若内外同时有电荷（导体接地或不接地），则将各情况叠加.

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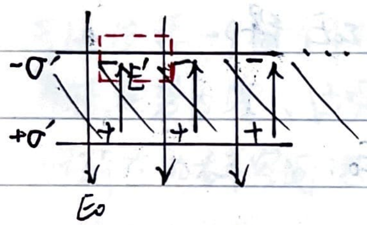
② 静电平衡时的电介质：

$\left\{\begin{array}{l}\text{极性分子} \to \text{固有电矩} \to \text{外电场处固有电矩排列较整齐.} \\ \text{非极性分子} \to \text{感生电矩} \to \text{外电场使正负电荷重心分开}\end{array}\right\}$ 宏观效果：产生面极化电荷.

设外加电场为 $\vec{E_0}$，极化电荷面密度 σ'，由它感应的场强为 $\vec{E'}$，电介质的相对介电常数为 ε_r，故空间中合场强 $\vec{E}=\vec{E_0}+\vec{E'}$，以长方体电介质为例：



由推广的高斯定律：$\varepsilon_0\oint_S(\varepsilon_r\vec{E})d\vec{S}=q$，q为自由电荷.

如图取一高斯面：$\because q=0$ $\therefore -\varepsilon_0E_0+\varepsilon_0\varepsilon_r(E_0-E')=0$

$\therefore E'=\frac{\varepsilon_r-1}{\varepsilon_r}E_0$. $\therefore$ 电介质内的合场强

$E=E_0-E'=\frac{E_0}{\varepsilon_r}$. 又由高斯定理：

$\varepsilon_0(-E_0\Delta S+E\Delta S)=-\sigma'\Delta S$ 解得：$-\sigma'=-\frac{\varepsilon_0(\varepsilon_r-1)}{\varepsilon_r}E_0$.

综上：由初始条件 E_0 以及 ε_r. 由高斯定理

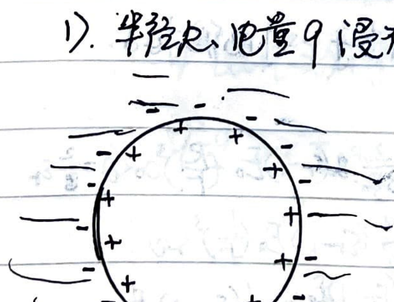
$\left\{\begin{array}{l}\varepsilon_0\oint_S\vec{E}\,d\vec{S}=q_{自由}+q_{极化} \\ \varepsilon_0\oint_S(\varepsilon_r\vec{E})d\vec{S}=q_{自由}.\end{array}\right\}$ 导体一般不是等势体，表面一般不是等势面，极化电荷分布于表面.

可解得

$E=\frac{E_0}{\varepsilon_r}$，$E'=\frac{\varepsilon_r-1}{\varepsilon_r}E_0$，$\sigma'=\frac{\varepsilon_0(\varepsilon_r-1)}{\varepsilon_r}\cdot E_0$.

定义 $\vec{P}=\varepsilon_0(\varepsilon_r-1)\vec{E}$ 为电极化强度，则 $\sigma'=\frac{\vec{P}\cdot\Delta\vec{S}}{\Delta S}=\vec{P}\cdot\vec{e}$.

1). 半径r、电量q 浸在 ε_r 的电介质中：



由 $\varepsilon_0\oint_S(\varepsilon_r\vec{E})d\vec{S}=q$ 得

$\varepsilon_0\varepsilon_rE\cdot4\pi r^2=q$

$\therefore E=\frac{q}{4\pi\varepsilon_0\varepsilon_r r^2}$.

由 $\varepsilon_0\oint_S\vec{E}\,d\vec{S}=q+q'$ 得

$q'=\varepsilon_0\cdot\frac{q}{4\pi\varepsilon_0\varepsilon_r r^2}\cdot4\pi r^2-q=\frac{q}{\varepsilon_r}-q=-\frac{\varepsilon_r-1}{\varepsilon_r}\cdot q$.

2). 两平行金属板 $(+\sigma_0、-\sigma_0)$ 间一半空间充以 ε_r 的电介质.

如下页图：

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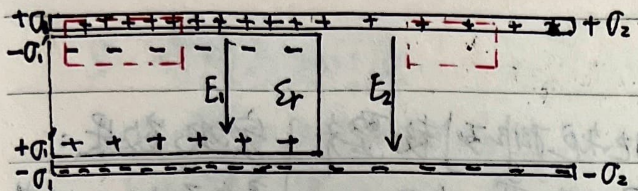


由电荷守恒 $\sigma_1+\sigma_2=2\sigma_0$

由高斯定理：

<1> $(\sigma_1\Delta S-\sigma_1'\Delta S)=\varepsilon_0E_1\Delta S$

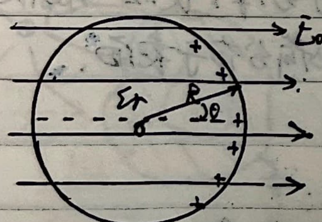
$\therefore \sigma_1-\sigma_1'=\varepsilon_0E_1$，<2> $\sigma_1\cdot\Delta S=\varepsilon_0\varepsilon_rE_1\Delta S$ $\therefore \sigma_1=\varepsilon_0\varepsilon_rE_1$

<3> $\sigma_2\Delta S=\varepsilon_0E_2\Delta S$ $\therefore \sigma_2=\varepsilon_0E_2$ 又$\because U_1=U_2$ $\therefore E_1=E_2$

故由 $\sigma_1+\sigma_2=2\sigma_0$，$\sigma_1-\sigma_1'=\varepsilon_0E_1$，$\sigma_1=\varepsilon_0\varepsilon_rE_1$，$\sigma_2=\varepsilon_0E_1$ 得

$$\left.\begin{aligned}\sigma_1&=\frac{2\varepsilon_r}{1+\varepsilon_r}\sigma_0\\ \sigma_2&=\frac{2}{1+\varepsilon_r}\sigma_0\\ \sigma_1'&=\frac{2(\varepsilon_r-1)}{1+\varepsilon_r}\sigma_0\end{aligned}\right\}\ E_1=E_2=\frac{2\sigma_0}{\varepsilon_0(1+\varepsilon_r)}=\frac{2}{1+\varepsilon_r}E_0.$$

3). 匀强场E_0中半径为R介电常数为ε_r的球。



极角θ处 $\vec{P}=\varepsilon_0(\varepsilon_r-1)\vec{E_0}$

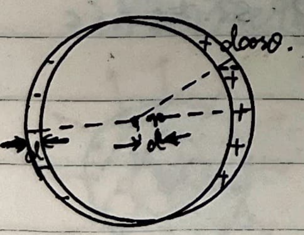
$\sigma'=\vec{P}\cdot\hat{S}=P\cos\theta=\varepsilon_0(\varepsilon_r-1)E_0\cos\theta$.

等效于两个体密度为ρ球心相距为d ($d\ll R$)的球

$\therefore q_总=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sigma'Rd\theta=\varepsilon_0(\varepsilon_r-1)E_0R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta$

$=2\varepsilon_0(\varepsilon_r-1)E_0R=2\sigma_0R$

$\rho=\frac{q}{d}=\frac{\varepsilon_0(\varepsilon_r-1)E_0}{d}$



而两个球心相距为d的球正相当于一对电偶极子。

$q_总'=\rho\cdot\frac{4}{3}\pi R^3$.

$\therefore$ 球外电场 $\begin{cases}E_r=\frac{1}{4\pi\varepsilon_0}\frac{2qd\cos\theta}{r^3}=\frac{2}{3}(\varepsilon_r-1)E_0(\frac{R}{r})^3\cos\theta\\ E_\theta=\frac{1}{4\pi\varepsilon_0}\frac{qd\sin\theta}{r^3}=\frac{1}{3}(\varepsilon_r-1)E_0(\frac{R}{r})^3\sin\theta.\end{cases}$

球内电场 $E=\frac{\rho}{3\varepsilon_0}=\frac{\rho d}{3\varepsilon_0}=\frac{\varepsilon_0(\varepsilon_r-1)E_0}{3\varepsilon_0}=\frac{1}{3}(\varepsilon_r-1)E_0$.

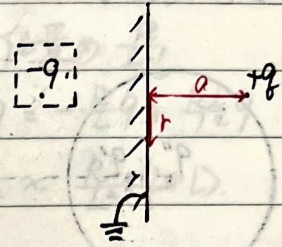
专题：电像法.

原理：把感应电荷产生的效果用一个或几个像点电荷取代。

本质：在原电荷所在空间内，空间场强、电势分布不变。

原因：唯一性定理。

1. 无限宽接地平板。



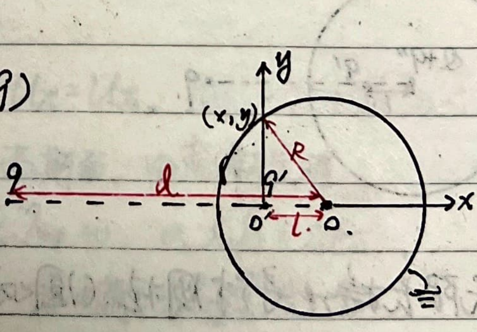
由于导体平板电势为0，电偶极子中垂面电势也为0。可以设想+q相对平板的像点处有一负电荷，保证平板电势为零，由唯一性定理，用-q代替接地平板，+q所在的右半空间电场不变。

∴平板上场强分布 $E=\frac{k\cdot 2aq}{(r^2+a^2)^{\frac{3}{2}}}$ 方向垂直于板

∴ $\sigma=\frac{aq}{2\pi(r^2+a^2)^{\frac{3}{2}}}$

∴平板上总电荷 $q'=\int_0^\infty \sigma\cdot 2\pi r\,dr=-q$，故平板左边空间无电场，+q受到平板上感应电荷的静电力 $F=\frac{kq^2}{(2a)^2}$，相当于-q对其的力。

2. 接地球壳。



设有一正电荷q、一负电荷q' $(-q'<q)$ 存在于空间中，则 $\varphi=0$ 的等势面为一球。

证明：∵ $U_0=0$ ∴ $\frac{kq'}{\sqrt{x^2+y^2}}+\frac{kq}{\sqrt{(d+l-x)^2+y^2}}=0$.

∴ $q'^2[(d+l-x)^2+y^2]=q^2(x^2+y^2)$

∴ $(q^2-q'^2)x^2+2(d-l)q'^2x+(q^2-q'^2)y^2=q'^2(d-l)^2$.

配方得 $(q^2-q'^2)[x-\frac{(d-l)q'^2}{q^2-q'^2}]^2+(q^2-q'^2)y^2=\frac{(d-l)^2q^2q'^2}{q^2-q'^2}$.

∵ $q'\neq q$ ∴ $[x-\frac{(d-l)q'^2}{q^2-q'^2}]^2+y^2=\frac{q^2q'^2(d-l)^2}{(q^2-q'^2)^2}$. 由对称性可知等势面是球

由此可知，一个导体球壳外有一点电荷，这一系统在球外产生的电场可由一q'替代。q'距球心 $l=\frac{(d-l)q'^2}{q^2-q'^2}$，球半径 $R=\frac{qq'(d-l)}{q^2-q'^2}$.

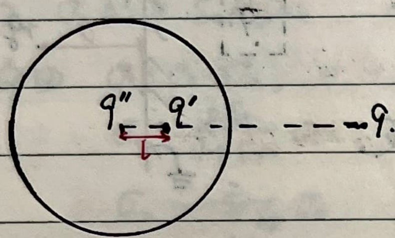
解得：$q'=-\frac{R}{d}q$，$l=\frac{R^2}{d}$

又∵ O点 $\varphi=0$，∴ $\frac{kq}{d}+\frac{kq''}{R}=0$ ∴ $q''=-\frac{R}{d}q=q'$

即球面上感应出的总电荷等于镜像电荷电量.

讨论：① 若开始时球壳不接地也不带电荷，在球壳外加一 $+q$.

1) 为保证球是等势体，在 $l=\frac{R^2}{d}$ 处加 $q'=-\frac{R}{d}q$，使球壳电势为零.

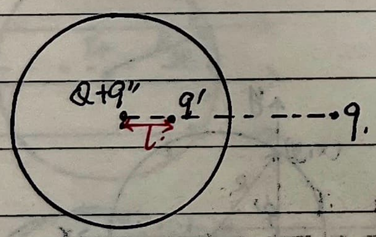


2) 由于加了 $+q$，球壳电势并不为零，为保证球壳为等势体，且净电荷为零，应在球壳表面均匀覆盖一层 $q''=-q'=\frac{R}{d}q$，而均匀覆于表面的 q'' 又等效于在球心处.

综上，$q'=-\frac{R}{d}q$，$x=\frac{R^2}{d}$；$q''=\frac{R}{d}q$，$x=0$.

②. 若开始时球壳不接地，且带电量为 Q_0（$\frac{kQ_0}{R}=U_0$），球壳外加一 $+q$.

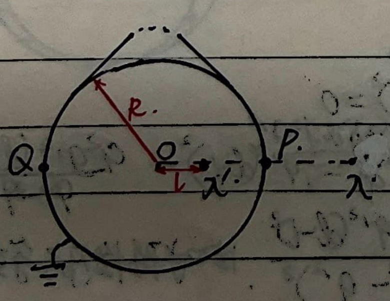
同理，在 $l=\frac{R^2}{d}$ 处加 $q'=-\frac{R}{d}q$ 使 $\varphi=0$



在球心处加 $q''=\frac{R}{d}q$ 使 $Q=0$，$\varphi=\frac{kq''}{R}$.

在球心处加 $q'''=Q_0$，使 $Q=Q_0$，$\varphi=\frac{k(q''+q''')}{R}$

3. 无限长接地导体圆柱.（圆心外 d 处有线密度 λ 的直线）



λ 在 P 点的电势 $\varphi_P=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d+R}{d-R}$.（以Q点为电势零点）

λ' 在 P 点的电势 $\varphi_P'=\frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{R+l}{R-l}$.

∴ $\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d+R}{d-R}+\frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{R+l}{R-l}=0$. <1>

λ 在 Q 点的电势 $\varphi_Q=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d-R}{d+R}$（以P点为电势零点）

λ' 在 Q 点的电势 $\varphi_Q'=\frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{R-l}{R+l}$.

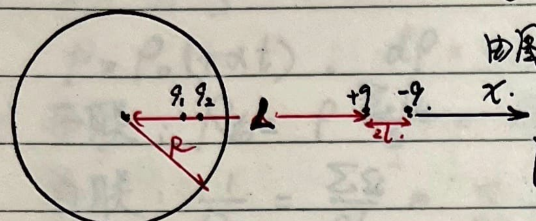
∴ $\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d-R}{d+R}+\frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{R-l}{R+l}=0$ <2>.

由<1><2>两式解得 $\lambda'=-\lambda$，$l=\frac{R^2}{d}$.

4. 电偶极子在不接地球壳前d处.

电矩 $P = q\cdot 2l$

由图: $+q$坐标$L+l$, $-q$坐标$L-l$.



$+q$像电荷 $q_1 = -\frac{R}{L+l}q$, 位置为 $\frac{R^2}{L+l}$.

$-q$像电荷 $q_2 = +\frac{R}{L-l}q$, 位置为 $\frac{R^2}{L-l}$.

$\because L \gg l$, $\therefore q_1 \approx -\frac{R}{L^2}(L-l)q = -\frac{R}{L}q + \frac{Rl}{L^2}q$.

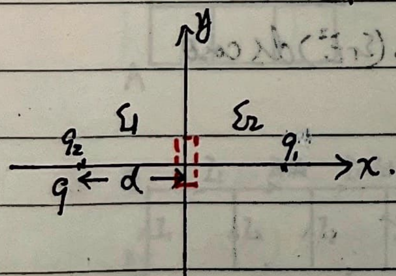
$x_1 \approx \frac{R^2}{L^2}(L-l)$; $q_2 \approx +\frac{R}{L^2}(L+l)q = \frac{R}{L}q + \frac{Rl}{L^2}q$, $x_2 \approx \frac{R^2}{L^2}(L+l)$.

$\therefore p' \approx \frac{R}{L}q \cdot \frac{2R^2}{L^2}l = \frac{R^3}{L^3}\cdot P$.

因在q_1, q_2中点处($\frac{R^2}{L}$)还有一像点电荷: $q' = \frac{q_1+q_2}{2} = \frac{Rl}{L^2}q$.

综上, 所有像电荷
$$\left\{\begin{array}{ll} q_1 = -\frac{R}{L}q + \frac{Rl}{L^2}q, & x_1 = \frac{R^2}{L} - \frac{R^2}{L^2}l \\ q_2 = \frac{R}{L}q + \frac{Rl}{L^2}q, & x_2 = \frac{R^2}{L} + \frac{R^2}{L^2}l \\ q' = \frac{Rl}{L^2}q, & x' = \frac{R^2}{L} \end{array}\right.$$
$p' \approx \frac{R^3}{L^3}\cdot P$.

5. 点电荷对无限大介质平面的镜像.



① 在介质平面处 $U_左 = U_右$.

② 在原点附近作一高斯面, 由高斯定理

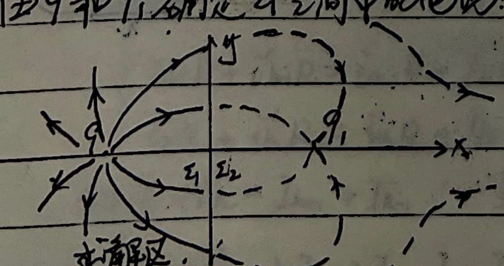
$\varepsilon_0\oint(\varepsilon_r \vec{E})\cdot d\vec{s} = \sum q_{内} = 0$. $\therefore \varepsilon_1 E_1 = \varepsilon_2 E_2$.

设 $x = \pm d$ 处各有一像电荷, 分别是 q_1、q_2.

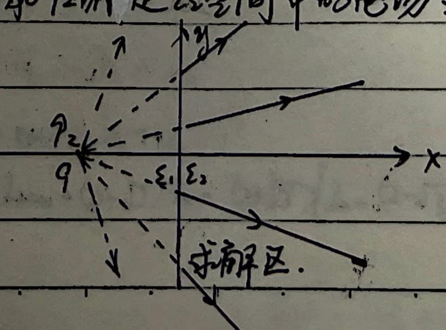
由①②条件, 在原点处有 $\frac{q+q_1}{4\pi\varepsilon_1 d} = \frac{q+q_2}{4\pi\varepsilon_2 d}$, $\varepsilon_1\frac{q-q_1}{4\pi\varepsilon_1 d^2} = \varepsilon_2\frac{q+q_2}{4\pi\varepsilon_2 d^2}$

由上两式解得 $q_1 = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$; $q_2 = -\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$

由q和q_1确定ε_1空间中的电场:



由q和q_2确定ε_2空间中的电场:



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专题：静电场中的虚功原理

1. 平行板电容器正极板所受静电力

① q不变时：

$$W=\frac{1}{2}\varepsilon_0E^2\cdot Sd$$
$$=\frac{1}{2}\varepsilon_0\left(\frac{Q}{Cd}\right)^2Sd$$
$$=\frac{1}{2}\frac{Q^2d}{\varepsilon_0 S}.$$

$$\therefore F=-\frac{\partial W}{\partial d}=-\frac{Q^2}{2\varepsilon_0 S}=-\frac{\sigma^2S}{2\varepsilon_0}=-\frac{1}{2}\varepsilon_0E^2\cdot S$$ 负号表示吸引力.

② U不变时：

$$W=\frac{1}{2}\varepsilon_0\left(\frac{U}{d}\right)^2Sd$$
$$=\frac{1}{2}\frac{\varepsilon_0U^2S}{d}$$

$$\therefore F=\frac{\partial W}{\partial d}=-\frac{\varepsilon_0U^2S}{2d^2}=-\frac{1}{2}\varepsilon_0E^2\cdot S$$ 负号表示吸引力.

综上，$|F|=\frac{1}{2}\varepsilon_0E^2S$，可推广到任何带电体。$\frac{dF}{dS}=-\frac{1}{2}\varepsilon_0E^2=-w_e$

$$F=\int_S P\,dS=\frac{1}{2\varepsilon_0}\int_S\sigma^2dS$$，且
$$W=\int_V\frac{1}{2}\varepsilon_0(\varepsilon_rE^2)dV$$
$$F=\int_S\frac{1}{2}\varepsilon_0(\varepsilon_rE^2)dS\cos\theta$$

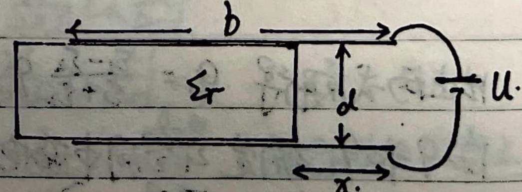
2. 带电导体球上下半球的相互作用力

$$F=\frac{1}{2\varepsilon_0}\int_S\sigma^2dS\cos\theta=\frac{\sigma^2}{2\varepsilon_0}\cdot\pi R^2$$

3. 平行金属板中拉电介质块的力

$$W=\frac{1}{2}\left(\frac{\varepsilon_0\varepsilon_r(b-x)l}{d}+\frac{\varepsilon_0xl}{d}\right)U^2$$

$$\therefore F=\frac{\partial W}{\partial x}=\frac{\varepsilon_0(\varepsilon_r-1)lU^2}{2d}$$



九. 电阻 电容 电感

1. 电阻.

① $R=\int\rho\frac{dl}{S}$

$\rho=\rho_0(1+\alpha t)$, $d\rho=\alpha\rho_0\,dt$. ∴$R=R_0(1+\alpha t)$.

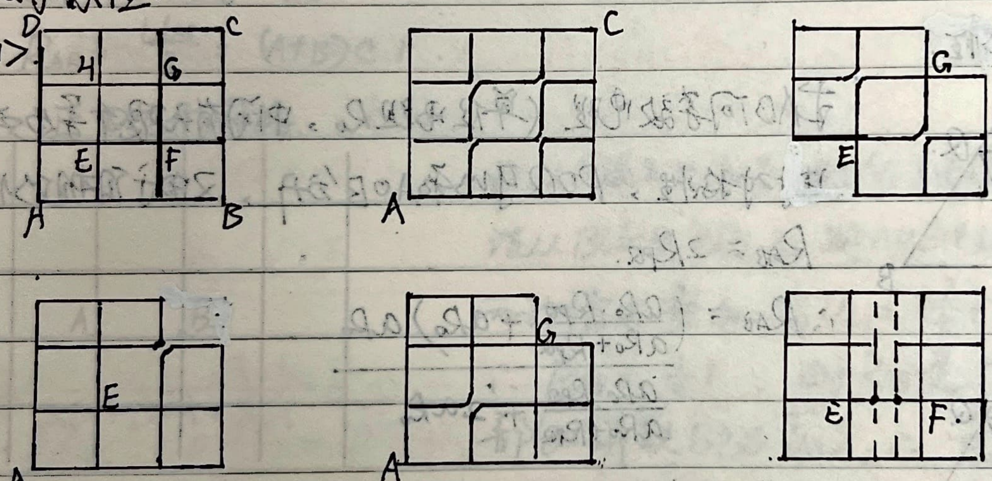
② 串联: $R_总=\rho\frac{\sum l_i}{S}=\sum\rho\frac{l_i}{S}=\sum R_i$; $R_总=\frac{U_总}{I}=\sum\frac{U_i}{I}=\sum R_i$.

并联: $\frac{1}{R_总}=\frac{\sum S_i}{\rho l}=\sum\frac{S_i}{\rho l}=\sum\frac{1}{R_i}$; $\frac{1}{R_总}=\frac{I_总}{U}=\sum\frac{I_i}{U}=\sum\frac{1}{R_i}$

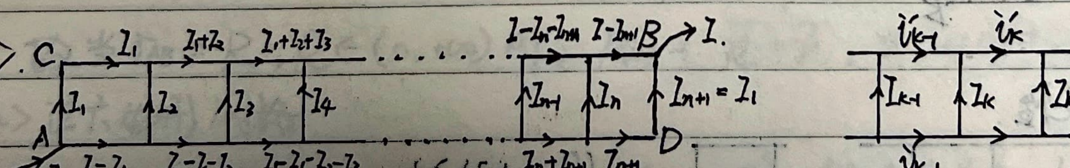
③ 等效电阻.

1) 对称性.

<1>



<2>



$U_{CB}=U_{AD}=[I_1+(I_1+I_2)+(I_1+I_2+I_3)+\cdots+(I-I_{n+1})]R$

$=[(I-I_1)+(I-I_1-I_2)+(I-I_1-I_2-I_3)+\cdots+I_{n+1}]R$.

∴$2U_{CB}=nIR$

又∵ $\begin{cases} I_{k-1}R+i_{k-1}'R=i_{k-1}R+I_kR \\ I_kR+i_k'R=i_kR+I_{k+1}R\end{cases}$ $\begin{cases} i_{k-1}'+I_k=i_k' \\ i_{k-1}=I_k+i_k\end{cases}$

解得 $4I_k=I_{k+1}+I_{k-1}$ 变形为 $I_{n+1}-(2-\sqrt3)I_n=(2+\sqrt3)[I_n-(2-\sqrt3)I_{n-1}]$.

∵共轭对称 ∴$I_{n+1}=I_1$, $I_n=I_2$.

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$$\therefore I_2 = \frac{(2+\sqrt{3})^{n-2}-(2-\sqrt{3})^{n-2}}{(2+\sqrt{3})^{n-1}-(2-\sqrt{3})^{n-1}+(2-\sqrt{3})-(2+\sqrt{3})} I_1$$

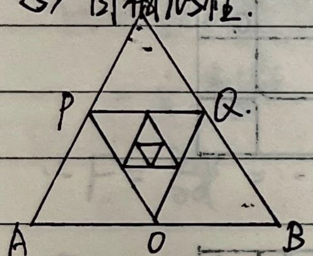
又∵在第1中：$2I_1R=(I-I_1)R \Longrightarrow I_2 = 3I_1 - I$.

由上两式得 $I_1 = \left[3-\frac{(2+\sqrt{3})^{n-2}-(2-\sqrt{3})^{n-2}}{(2+\sqrt{3})^{n-1}-(2-\sqrt{3})^{n-1}-2\sqrt{3}}\right]^{-1}\cdot I$

$\therefore U_{AB} = I_1R + U_{BC}$

$$R_{AB} = \frac{U_{AB}}{I} = \left\{\frac{n}{2}+\left[3-\frac{(2+\sqrt{3})^{n-2}-(2-\sqrt{3})^{n-2}}{(2+\sqrt{3})^{n-1}-(2-\sqrt{3})^{n-1}-2\sqrt{3}}\right]^{-1}\right\}R.$$

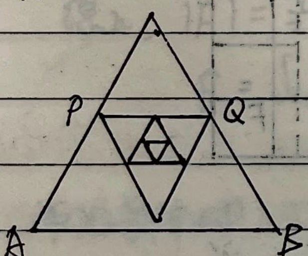
(3) 自相似性.



求AB间等效电阻（单位电阻R_0，中间有无限个等边三角形）

由于对称性，POQ可以和AOB分开，又由于自相似性，

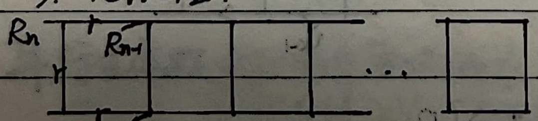
$R_{AB}=2R_{PQ}$.



$$\therefore R_{AB} = \frac{\left(\frac{aR_0\cdot R_{PQ}}{aR_0+R_{PQ}} + aR_0\right)aR_0}{\frac{aR_0\cdot R_{PQ}}{aR_0+R_{PQ}}+2aR_0}$$

$$\therefore R_{AB}=\frac{1}{3}(\sqrt{7}-1)aR_0$$

2). 无限性.



$$\frac{(R_{n-1}+2r)r}{R_{n-1}+3r}=R_n.$$

$$\therefore \frac{(2-\sqrt{3})r[R_{n-1}+(1-\sqrt{3})r]}{R_{n-1}+3r} = R_n+(1-\sqrt{3})r.$$

$$\therefore \frac{1}{(2-\sqrt{3})r} + \frac{(2+\sqrt{3})r}{(2-\sqrt{3})r[R_{n-1}+(1-\sqrt{3})r]} = \frac{1}{R_n+(1-\sqrt{3})r}$$

$$\therefore \frac{\frac{1}{R_n+(1-\sqrt{3})r}+\frac{1}{2\sqrt{3}r}}{\frac{1}{R_{n-1}+(1-\sqrt{3})r}+\frac{1}{2\sqrt{3}r}} = 7+4\sqrt{3}.$$

解得 $R_n = \dfrac{1}{-\frac{1}{2\sqrt{3}r} + \left[\frac{1}{R_1+(1-\sqrt{3})r} + \frac{1}{2\sqrt{3}r}\right](7+4\sqrt{3})^{n-1}} + (\sqrt{3}-1)r$.

当 $n \to \infty$ 时

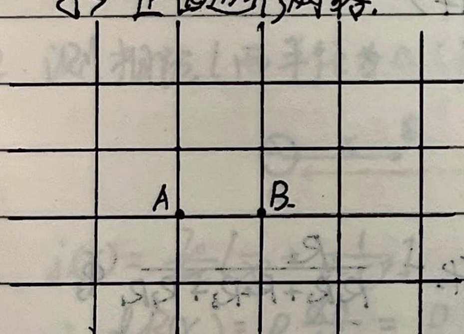
$R_n = (\sqrt{3}-1)r$.

3) 电流叠加法.

设流入电流为 I，算出所求支路 AB 中流过的电流 I_{AB}；设另一点流出的电流为 I，算出 AB 中电流 I'_{AB}。若 r_{AB} 是连接 AB 两点的导线的电阻且 $r_{AB}=Cr$，且 $I_{AB}=AI$，$I'_{AB}=BI$。 ∴ $U_{AB}=(I_{AB}+I'_{AB})r_{AB}=(A+B)C\cdot r\cdot I$

∴ $R_{AB}=\dfrac{U_{AB}}{I}=(A+B)C\cdot r$.

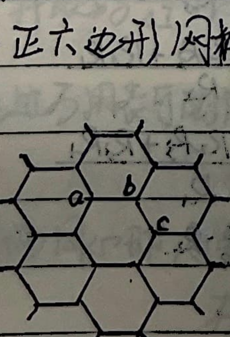
<1> 正四边形网格.



设从 A 流入电流 I，则 AB 上电流为 $\frac{I}{4}$。
设从 B 流出电流 I，则 AB 上电流为 $\frac{I}{4}$。
两种情况叠加，有 $U_{AB}=(\frac{I}{4}+\frac{I}{4})\cdot r$
∴ $R_{AB}=\frac{1}{2}r$。由于 R_{AB} 是 r_{AB} 和其余部分的并联(记为 $\overline{r_{AB}}$)，∴ $\frac{1}{r_{AB}}+\frac{1}{\overline{r_{AB}}}=\frac{2}{r}$
而 $r_{AB}=r$，∴ $\overline{r_{AB}}=r$.

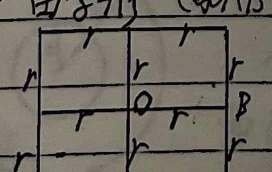
故当 $r_{AB}=R$，$R\in(0,+\infty)$ 时，$R_{AB}=\dfrac{1}{\frac{1}{R}+\frac{1}{r}}$；当 $r_{AB}=0$ 时，$R_{AB}=\overline{r_{AB}}=r$.

<2> 正六边形网格.



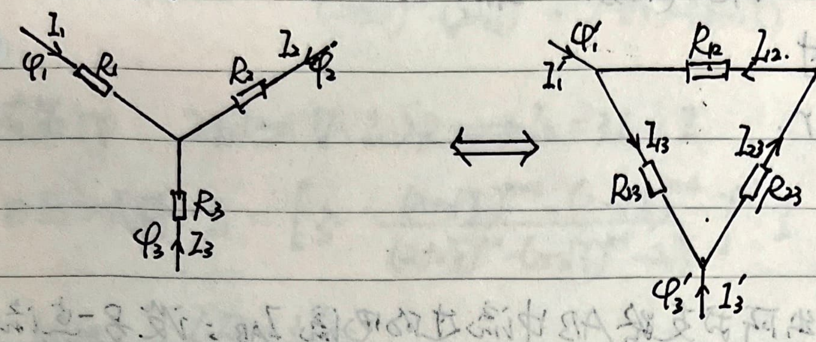
设从 a 流入电流 I，则 ab 上电流为 $\frac{I}{3}$，bc 上电流为 $\frac{I}{6}$；设从 c 流出电流 I，则 bc 上电流为 $\frac{I}{3}$，ab 上电流为 $\frac{I}{6}$。两种情况叠加，有
$U_{ac}=(\frac{I}{6}+\frac{I}{3})r+(\frac{I}{6}+\frac{I}{3})r=Ir$，∴ $R_{ac}=r$

<3> 田字形 (求AB的等效电阻)



设从 A 流入从 O 流出，解得 $I_1=\frac{I}{2}$，$I_2=\frac{I}{8}$
设从 O 流入从 B 流出，解得 $I_1''=\frac{I}{24}$，$I_2''=\frac{5I}{24}$
∴ $I_1=\frac{I}{2}+\frac{I}{24}=\frac{13}{24}I$；$I_2=\frac{I}{8}+\frac{5I}{24}=\frac{1}{3}I$
∴ $U_{AB}=I_1r+I_2\cdot 2r$；∴ $R_{AB}=\frac{U_{AB}}{I}=(\frac{13}{24}+\frac{2}{3})r=\frac{29}{24}r$

4) Y-Δ变换



等效本质：$\begin{cases} I_1=I_1',\ I_2=I_2',\ I_3=I_3' \\ \varphi_1=\varphi_1',\ \varphi_2=\varphi_2',\ \varphi_3=\varphi_3' \end{cases}$

Y形中有 $\begin{cases} \varphi_1-I_1R_1+I_3R_3=\varphi_3\ ;\ \varphi_1-I_1R_1+I_2R_2=\varphi_2 \ \text{(数电压法)} \\ I_1+I_2+I_3=0 \ \text{(基尔霍夫电流定律)} \end{cases}$

Δ形中有 $\begin{cases} \varphi_1'-I_{13}R_{13}=\varphi_3'\ ;\ \varphi_1'+I_{12}R_{12}=\varphi_2' \\ I_1'=I_{13}-I_{12} \end{cases}$

$\because I_1=I',\ \varphi_1=\varphi',\ \varphi_2=\varphi_2',\ \varphi_3=\varphi_3'$

而 $I_1=\dfrac{R_2+R_3}{R_1R_2+R_2R_3+R_3R_1}\varphi_1-\dfrac{R_3}{R_1R_2+R_2R_3+R_3R_1}\varphi_2-\dfrac{R_2}{R_1R_2+R_2R_3+R_3R_1}\varphi_3$

$I_1'=(\dfrac{1}{R_{31}}+\dfrac{1}{R_{12}})\varphi_1'-\dfrac{1}{R_{12}}\varphi_2'-\dfrac{1}{R_{31}}\varphi_3'$

$\therefore \dfrac{R_2+R_3}{R_1R_2+R_2R_3+R_3R_1}=\dfrac{1}{R_{31}}+\dfrac{1}{R_{12}}$ 解得 $R_{12}=\dfrac{R_1R_2+R_2R_3+R_3R_1}{R_3}$

$\dfrac{R_3}{R_1R_2+R_2R_3+R_3R_1}=\dfrac{1}{R_{12}}$ $\qquad R_{31}=\dfrac{R_1R_2+R_2R_3+R_3R_1}{R_2}$

$\dfrac{R_2}{R_1R_2+R_2R_3+R_3R_1}=\dfrac{1}{R_{31}}$ 由对称性 $R_{23}=\dfrac{R_1R_2+R_2R_3+R_3R_1}{R_1}$

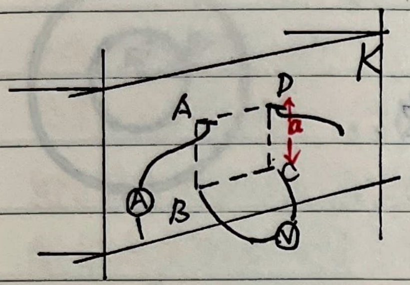
$\therefore \dfrac{R_{12}}{R_{31}}=\dfrac{R_2}{R_3}\Rightarrow R_3=\dfrac{R_{31}}{R_{12}}R_2,\ \dfrac{R_{31}}{R_{23}}=\dfrac{R_1}{R_2}\Rightarrow R_1=\dfrac{R_{31}}{R_{23}}\cdot R_2$

$\therefore R_{31}=\dfrac{R_{31}}{R_{23}}\cdot R_2+\dfrac{R_{31}}{R_{12}}R_2+\dfrac{R_{31}}{R_{12}}\cdot\dfrac{R_{31}}{R_{23}}\cdot R_2$

解得 $\left.\begin{aligned} R_2&=\dfrac{R_{23}R_{12}}{R_{12}+R_{23}+R_{31}} \\ \text{由对称性}\ R_3&=\dfrac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \\ R_1&=\dfrac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}} \end{aligned}\right\}$

专题：测电阻率.

1. 测无限大物质电阻率.



在K上划出一正方形ABCD.

在A处注入电流，D处输出同样强度的电流(I_0).

对注入A的电流，有 $j=\frac{I_0}{2\pi r^2}$，∴ $E=\frac{j}{\sigma}=\frac{I_0\rho}{2\pi r^2}$.

∴ $\varphi=-\int E\,dr=\frac{I_0\rho}{2\pi r}$

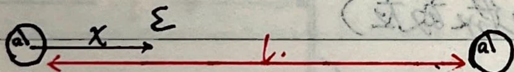
∴ $U_{BC}=\frac{I_0\rho}{2\pi a}-\frac{I_0\rho}{2\sqrt{2}\pi a}=\frac{(2-\sqrt{2})I_0\rho}{4\pi a}$.

同样，只考虑D输出的电流时，$U'_{BC}=\frac{(2-\sqrt{2})I_0\rho}{4\pi a}$，利用电流叠加原理.

$$U_0=U_{BC}+U'_{BC}=\frac{(2-\sqrt{2})I_0\rho}{2\pi a}.$$

∴ $\rho=(2+\sqrt{2})\pi a\frac{U_0}{I_0}$.

2. 测相距 l 两半径为 a ($l\gg a$) 金属球所在空间电阻率(万用表测得两球间R)



$j(x)=\frac{I_0}{4\pi}\left[\frac{1}{x^2}+\frac{1}{(l-x)^2}\right]$ $(a<x<l-a)$

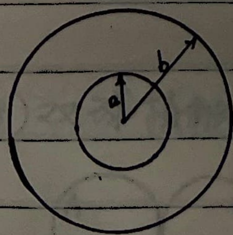
∴ $dR(x)=\rho\frac{dx}{\Delta S}=\rho\frac{dx}{\Delta I}j(x)=\frac{dx}{\Delta I}\cdot\frac{I_0\rho}{4\pi}\left[\frac{1}{x^2}+\frac{1}{(l-x)^2}\right]$

∴ $R=\frac{I_0\rho}{4\pi\Delta I}\int_a^{l-a}\left[\frac{1}{x^2}+\frac{1}{(l-x)^2}\right]dx=\frac{I_0\rho}{4\pi\Delta I}\cdot2\left(\frac{1}{a}-\frac{1}{l-a}\right)=\frac{I_0\rho}{2\pi\Delta I}\left(\frac{1}{a}-\frac{1}{l-a}\right)$.

∴并联的 $\frac{I_0}{\Delta I}$ 根导线的总电阻为 $\frac{R}{I_0/\Delta I}=\frac{\rho}{2\pi}\left(\frac{1}{a}-\frac{1}{l-a}\right)=R_总$.

通过万用表可测得 $R_总$，∴ $\rho=\frac{2\pi R_总 a(l-a)}{l-2a}$ 当 $l\to\infty$ 时 $\rho=2\pi R_总 a$

3. 测两同心球间物质的电阻率.



在两球面间取一半径为r的高斯面.有

$\oint\vec{E}\cdot d\vec{S}=E\cdot4\pi r^2=\frac{Q}{\varepsilon}=\rho\oint\vec{j}\cdot d\vec{S}=\rho\cdot I$

∴ $Q=\varepsilon\rho I$

而 $U_{ab}=\frac{Q}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)=IR$ ∴ $\frac{Q}{\varepsilon I}=\frac{4\pi R\cdot ab}{b-a}$

∴ $\rho=\frac{4\pi Rab}{b-a}$

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2. 电容.

① $C=\frac{Q}{U}$

$\begin{cases} U：两极间电势差 \\ Q：用导线将两极相连后通过导线的电量. \end{cases}$

② 串联：$\frac{1}{C_{总}}=\frac{U_{总}}{Q}=\sum\frac{U_i}{Q}=\sum\frac{1}{C_i}$.

并联：$C_{总}=\frac{Q_{总}}{U}=\sum\frac{Q_i}{U}=\sum C_i$

③ 电容器的能量：

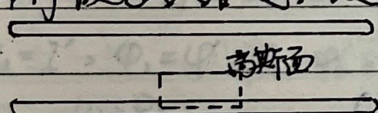
$$W=\int q\,dU=\int_0^Q q\frac{dq}{C}=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}\frac{Q^2}{\varepsilon S}\cdot d=\frac{1}{2}\left(\frac{Q}{\varepsilon S}\right)^2\varepsilon\cdot Sd$$

$$=\frac{1}{2}E^2\varepsilon Sd=\frac{1}{2}\varepsilon E^2\cdot V$$

能量密度 $w=\frac{dW}{dV}=\frac{1}{2}\varepsilon E^2$.

④ 典型电容器电容.

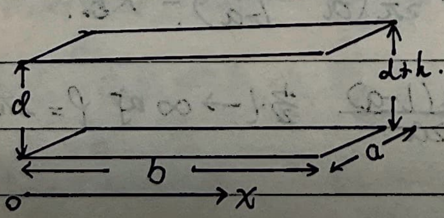
1) 平行板电容器（忽略边缘效应）



$E\Delta S=\frac{\sigma\Delta S}{\varepsilon_0}$, ∴ $E=\frac{\sigma}{\varepsilon_0}=\frac{U}{d}=\frac{q}{\varepsilon_0 S}$

∴ $C=\frac{q}{U}=\frac{\varepsilon_0 S}{d}$.

2) 平行板倾斜电容器（$h\ll d$）

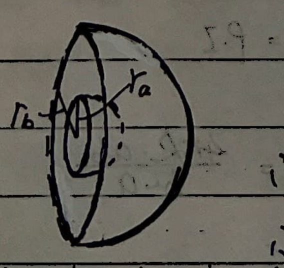


将整个电容器看成无数dC并联而成.

$$dC=\frac{\varepsilon_0\,a\,dx}{d+\frac{h}{b}x}$$

$$∴ C=\int_0^b\frac{\varepsilon_0\,a\,dx}{d+\frac{h}{b}x}=\frac{\varepsilon_0 ad}{h}\ln\left(1+\frac{h}{d}\right).$$

3) 球形电容器.



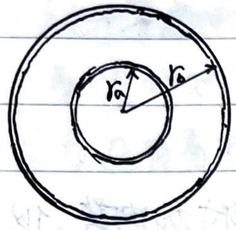
$\varphi_A=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a}+\frac{-q}{r_b}\right)$

$\varphi_B=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_b}+\frac{-q}{r_b}\right)$

∴ $U_{AB}=\varphi_A-\varphi_B=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$.

∴ $C=\frac{q}{U_{AB}}=\frac{4\pi\varepsilon_0 r_a r_b}{r_b-r_a}$.

4) 圆柱形电容器（长为l）



在两圆柱中间取一圆柱体高斯面，半径为r，有

$E\cdot 2\pi r\cdot l = \frac{q}{\varepsilon_0}$，∴ $E=\frac{q}{2\pi\varepsilon_0 r l}$

∴ $U_{ab}=\int_a^b E\,dr = \frac{q}{2\pi\varepsilon_0 l}\int_a^b \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 l}\ln\frac{r_b}{r_a}$

∴ $C=\frac{q}{U_{ab}}=\frac{2\pi\varepsilon_0 l}{\ln\frac{r_b}{r_a}}$，当 $r_b-r_a \ll r_a, r_b$ 时

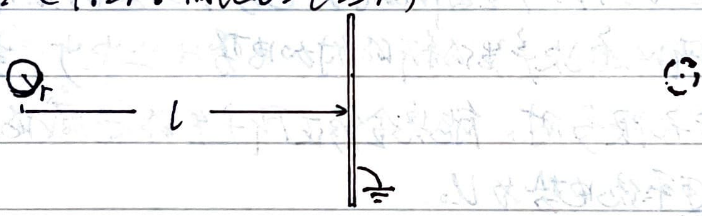
$C\approx \frac{2\pi\varepsilon_0 r_a\cdot l}{r_b-r_a}$

5) 孤立导体球：将无限远处看成另一极板.

∵ $\varphi_\infty=0$，$\varphi_R=\frac{Q}{4\pi\varepsilon_0 R}$

∴ $U=\frac{Q}{4\pi\varepsilon_0 R}$ ∴ $C=\frac{Q}{U}=4\pi\varepsilon_0 R$.

6) 球板电容器（半径r，相距l，$l\gg r$）



在金属板右边作出金属球的像，撤去金属板，在此双球系统中

$U_{ab}=\frac{kq}{r}-\left(-\frac{kq}{r}\right)=\frac{q}{2\pi\varepsilon_0 r}$

∴ $C=\frac{q}{U_{ab}}=2\pi\varepsilon_0 r$

此双球系统恰相当于两个球板电容器串联，故球板电容器电容

$C'=4\pi\varepsilon_0 r$

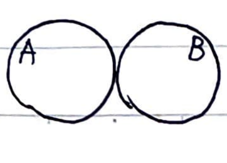
7) 双球相连电容器（半径r_a、r_b，相距l，$l\gg r_a, r_b$）

∵两球用导线相连，∴ $\varphi_a=\varphi_b \Rightarrow \frac{kq_1}{r_a}=\frac{kq_2}{r_b}$.

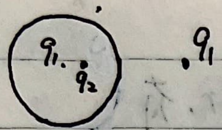
以无限远处为另一极，$\varphi_\infty=0$ ∴ $U=\varphi_a-\varphi_\infty=\frac{kq_1}{r_a}$

∴ $C=\frac{q_1+q_2}{U}=\frac{q_1+\frac{r_b}{r_a}q_1}{\frac{kq_1}{r_a}}=4\pi\varepsilon_0(r_a+r_b)$.

8) 双球接触电容器（半径R，相距$2R$）



设系统电势为U_0. 设想两球电荷集中在球心，不计相互感应时，两球电势均为U_0，且$q_1=4\pi\varepsilon_0 R U_0$.

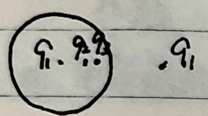


由于相互感应会使两球电势比 U_0 偏大，为消除附加电势需在各球内加一个像电荷

$$q_2 = -\frac{R}{2R}q_1 = -\frac{1}{2}q_1$$

距各球心距离 $x_2 = \frac{R^2}{2R} = \frac{1}{2}R$

这样，各自内部的像电荷可消除对方的附加电势，但又使自己产生新的附加电势，需要新的像电荷去抵消。

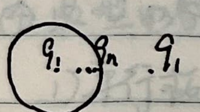


$$q_3 = -\frac{R}{2R-x_2}q_2 = \frac{1}{3}q_1$$

$$x_3 = \frac{R^2}{2R-x_2} = \frac{2}{3}R.$$

⋮

以此类推，每次新加的像电荷能抵消上次产生的附加电势，而又产生新的附加电势，又因为 $|q_{n+1}|<|q_n|$，所以每次产生的新的附加电势比上次少，当像电荷趋于无限多时，能完全修正所产生的递减的附加电势，使系统电势为 U_0.



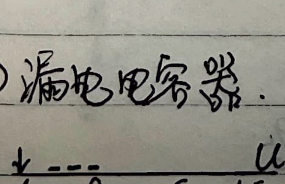
即 $\begin{cases} q_n = \frac{(-1)^{n-1}}{n}q_1 \\ x_n = \frac{n-1}{n}R \end{cases}$

又由导体球上的总电量就等于 q_1 加像电荷的总电量，

$$Q_{总} = 2(q_1+q_2+\cdots) = 2q_1(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots) = 8\pi\varepsilon_0 R U_0 \ln 2$$

$$\therefore C = \frac{Q_{总}}{U_0} = 8\pi\varepsilon_0 R\ln 2.$$

9) 漏电电容器.



由于两极间加了电压 U，$I = \frac{U}{\rho_1\frac{d_1}{S}+\rho_2\frac{d_2}{S}} = \frac{US}{\rho_1 d_1+\rho_2 d_2}$

$\therefore E_1 = \rho_1\frac{I}{S} = \frac{\rho_1 U}{\rho_1 d_1+\rho_2 d_2}$，$E_2 = \rho_2\frac{I}{S} = \frac{\rho_2 U}{\rho_1 d_1+\rho_2 d_2}$

又 $\because C_1 = \frac{\varepsilon_0\varepsilon_1 S}{d_1}$，$C_2 = \frac{\varepsilon_0\varepsilon_2 S}{d_2}$.

$\therefore q_1 = E_1 d_1 C_1 = \frac{\rho_1\varepsilon_0\varepsilon_1 SU}{\rho_1 d_1+\rho_2 d_2}$. $q_2 = E_2 d_2 C_2 = \frac{\rho_2\varepsilon_0\varepsilon_2 SU}{\rho_1 d_1+\rho_2 d_2}$

$\therefore$ 两介质交界面上自由电荷面密度

$$\sigma = \frac{\varepsilon_0(\rho_1\varepsilon_1-\rho_2\varepsilon_2)U}{\rho_1 d_1+\rho_2 d_2}$$

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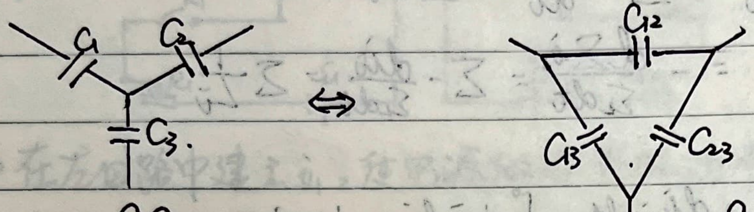
⑤ 等效电容

类似于等效电阻，有对称性、无限性等解法。又由于二端无源电路中，R 与 $\frac{1}{C}$ 的地位相同。（串联：$R=\sum R_i$，$C=(\sum \frac{1}{C_i})^{-1}$；并联：$R=(\sum \frac{1}{R_i})^{-1}$，$C=\sum C_i$）

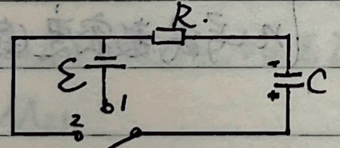
故与 R 有着相反的 Y－Δ 变换.



$$\begin{cases} C_{12}=\dfrac{C_1C_2}{C_1+C_2+C_3} \\ C_{23}=\dfrac{C_2C_3}{C_1+C_2+C_3} \\ C_{31}=\dfrac{C_3C_1}{C_1+C_2+C_3} \end{cases}$$

$$\begin{cases} C_3=\dfrac{C_{12}C_{23}+C_{12}C_{31}+C_{23}C_{31}}{C_{12}} \\ C_1=\dfrac{C_{12}C_{23}+C_{12}C_{31}+C_{23}C_{31}}{C_{23}} \\ C_2=\dfrac{C_{12}C_{23}+C_{12}C_{31}+C_{23}C_{31}}{C_{13}} \end{cases}$$

⑥ RC 电路.



1) 充电：$\varepsilon - iR - \frac{q}{C}=0$，$i=\frac{dq}{dt}$

$\therefore \varepsilon = R\frac{dq}{dt}+\frac{q}{C} \Rightarrow dt=\frac{RC\,dq}{\varepsilon C - q}$

$\therefore \int_0^t dt = -RC\int_0^q \frac{d(C\varepsilon - q)}{C\varepsilon - q} \Rightarrow t=-RC\ln\frac{C\varepsilon - q}{C\varepsilon}$.

解得 $q=C\varepsilon(1-e^{-\frac{t}{RC}})$，$i=\frac{dq}{dt}=\frac{\varepsilon}{R}e^{-\frac{t}{RC}}$.

2) 放电：$iR=\frac{q}{C}$，$i=-\frac{dq}{dt}$

$\therefore R\frac{dq}{dt}+\frac{q}{C}=0 \Rightarrow dt=-RC\frac{dq}{q}$

$\therefore t=-RC\ln\frac{q}{q_0}=-RC\ln\frac{q}{C\varepsilon}$，$\therefore q=C\varepsilon e^{-\frac{t}{RC}}$

$i=-\frac{dq}{dt}=\frac{C\varepsilon}{RC}e^{-\frac{t}{RC}}=\frac{\varepsilon}{R}e^{-\frac{t}{RC}}$. 与充电的电流曲线一样.

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3. 电感

① $\varepsilon_L = -\frac{d\psi}{dt} = -N\frac{d\phi}{dt} = -\frac{d(Li)}{dt} = -L\frac{di}{dt}$.

∴ $L = -\varepsilon_L\frac{dt}{di}$.

② 串联：$L_{总} = -\varepsilon_{总}\frac{dt}{di} = \sum \varepsilon_i\frac{dt}{di} = \sum L_i$.

并联：$\frac{1}{L_{总}} = -\frac{di_{总}}{\varepsilon_i dt} = -\frac{d\sum i_i}{\varepsilon_i dt} = \sum -\frac{di_i}{\varepsilon_i dt} = \sum\frac{1}{L_i}$

③ 电感的能量：

$W = \int \varepsilon i dt = \int -L\frac{di}{dt} i dt = \int_I^0 -Li\,di = \frac{1}{2}LI^2$.

$= \frac{1}{2}\mu_0 n^2 V I^2$ 又由 $B=\mu_0 nI$ 消去 nI. 得

$W = \frac{B^2}{2\mu_0}V$

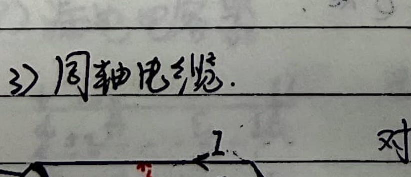
能量密度 $\omega = \frac{dW}{dV} = \frac{1}{2}\frac{B^2}{\mu_0}$.

④ 自感：

1) 螺线环：$\psi = N\phi = l\cdot n\cdot BS = ln\mu_0 nIS = L\cdot I$

∴ $L = \frac{\psi}{I} = \mu_0 n^2 S\cdot l = \mu_0 n^2 V$. [n: 导线数密度 (条/m).]

2) 环式螺线环：



对整个环用安培环路定理：

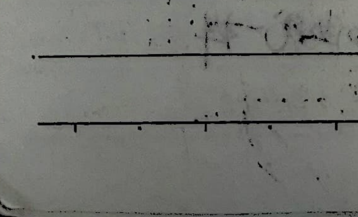
$B\cdot 2\pi r = \mu_0 NI$

∴ $B = \frac{\mu_0 NI}{2\pi r}$

∴ 磁通匝链数 $\psi = \int_a^b B\cdot h\cdot dr = \frac{\mu_0 NIh}{2\pi}\ln\frac{b}{a}$

∴ $L = \frac{\mu_0 Nh}{2\pi}\ln\frac{b}{a}$

3) 同轴电缆.



对单位长度的电缆，有

$\psi = \int_a^b Bl\,dr = \frac{\mu_0 Il}{2\pi}\int_a^b\frac{dr}{r} = \frac{\mu_0 Il}{2\pi}\ln\frac{b}{a}$

∴ $L = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$.

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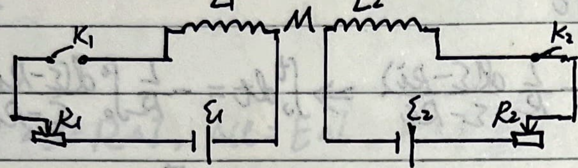
⑤ 互感

$\varepsilon_{21} = -M_{21}\frac{di_1}{dt}$ ， $\varepsilon_{12} = -M_{12}\frac{di_2}{dt}$

且 $M_{12} = M_{21} = M$

证明：



〈1〉在左回路中建立 i_1，使电源做功，在磁场中储能 $W_1 = \frac{1}{2}L_1 I_1^2$.

〈2〉在右回路中建立 i_2，使电源 ε_2 做功，储能 $W_2 = \frac{1}{2}L_2 I_2^2$，但在此过程中 L_1 产生反电动势，为保持 i_1 不变，ε_1 还应做功 $W_{12} = -\int \varepsilon_{12} I_1 dt$

$= \int M_{12} I_1 \frac{di_2}{dt} \cdot dt = M_{12} I_1 \int_0^{I_2} di_2 = M_{12} I_1 I_2$.

〈3〉经上述两过程后系统达到电流分别是 I_1 和 I_2 的状态，

$W_{总} = W_1 + W_2 + W_{12} = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$.

〈4〉从在右回路中建立 i_2 起达到同样的结果，做功同理为：

$W_{总} = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$

$\therefore M_{12} = M_{21}$

长螺线管与圆环的互感：（$l_管 \gg l_环$，环在管轴线上，轴与环平面垂直）.

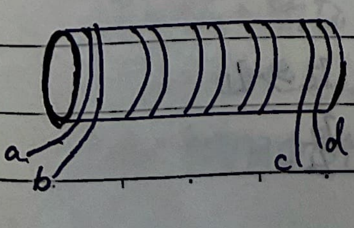
设螺线管内电流为 i_1，则通过圆环的全磁通 $\varphi = B_1 \pi r^2 = \pi r^2 \cdot \mu_0 n i_1$

$\therefore M_{21} = \pi r^2 \mu_0 n$

$\therefore M_{12} = M_{21} = \pi r^2 \mu_0 n$.

⑥ 等效电感

在互相远离的情况下，与等效电阻的计算方法类似；互相影响下的特例：



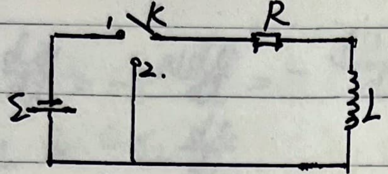
a-c 或 b-d 单独存在时自感为 L_0

a连b，cd接入电路： $L = 0$

c连b，ad接入电路：$\psi = (2N_0)(\mu_0 \cdot 2n_0 I) \cdot S$，$L = 4L_0$.

a连b、c连d：$\psi = N_0 \cdot (\mu_0 n_0 \frac{I}{2}) \cdot S \times 2$，$L = L_0$.

⑦ RL电路:



1) $\varepsilon + (-L\frac{di}{dt}) - iR = 0$

$\therefore dt = \frac{L\,di}{\varepsilon - Ri} = -\frac{L}{R}\frac{d(\varepsilon - Ri)}{\varepsilon - Ri} \Rightarrow \int_0^t dt = -\frac{L}{R}\int_0^i \frac{d(\varepsilon - Ri)}{\varepsilon - Ri}$

$\therefore t = -\frac{L}{R}\ln(\frac{\varepsilon - Ri}{\varepsilon})$ 解得 $i = \frac{\varepsilon}{R}(1 - e^{-\frac{Rt}{L}})$

$\therefore u_L = -L\frac{di}{dt} = -\varepsilon \cdot e^{-\frac{Rt}{L}}$

2) $-L\frac{di}{dt} - iR = 0$

$\therefore dt = -\frac{L}{R}\frac{di}{i} \Rightarrow \int_0^t dt = -\frac{L}{R}\int_{\frac{\varepsilon}{R}}^i \frac{di}{i}$

$\therefore t = -\frac{L}{R}\ln\frac{i}{\varepsilon/R}$ 解得 $i = \frac{\varepsilon}{R}e^{-\frac{Rt}{L}}$

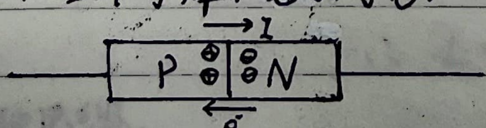
$\therefore u_L = -L\frac{di}{dt} = \varepsilon e^{-\frac{Rt}{L}}$

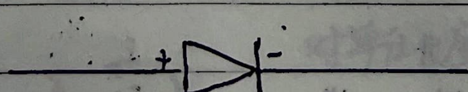
4. 半导体.

① 二极管.

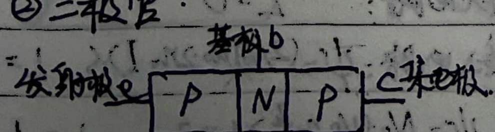
$\begin{cases} \text{理想二极管: 正向导通时 } R=0\text{, 反向 } R \text{ 为 } \infty. \\ \text{实际二极管: } i = I_s(e^{\frac{eu}{kT}} - 1). \end{cases}$

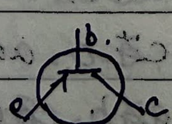
$\begin{cases} \text{P型半导体: 空穴导电.} \\ \text{N型半导体: 电子导电.} \end{cases}$





② 三极管





$\begin{cases} I_e = I_b + I_c\text{, } I_c \gg I_b \\ \beta = \frac{dI_c}{dI_b} \approx \frac{I_c}{I_b} \end{cases}$

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十. 稳恒电流.

1. 电流连续性方程：$\oint_S \vec{j}\,d\vec{S} = -\frac{dq}{dt}$.

稳恒条件：$\oint_S \vec{j}\,d\vec{S} = -\frac{dq}{dt} = 0$

2. 欧姆定律

$U = IR$

$\because \Delta I = j\Delta S$，$\Delta U = R\Delta I$，$\Delta U = E\Delta l$，$R = \rho\frac{\Delta l}{\Delta S}$

$\therefore j = \frac{\Delta I}{\Delta S} = \frac{\Delta U}{R\Delta S} = \frac{E\Delta l}{R\Delta S} = \frac{E\Delta l}{\rho\frac{\Delta l}{\Delta S}\cdot\Delta S} = \sigma E$.

写成矢量形式 $\vec{j} = \sigma\vec{E}$，代入高斯定律，$\frac{1}{\sigma}\oint_S \vec{j}\,d\vec{S} = \frac{q}{\varepsilon_0}$

$\therefore -\frac{dq}{dt} = \frac{\sigma}{\varepsilon_0}q$ 解得 $q = q_0 e^{-\frac{\sigma}{\varepsilon_0}t}$，表示在电路中建立稳恒电流的过程.

3. 全电路欧姆定律、焦耳定律.

① 对于纯电阻（电流做功全用于发热）电路，电源电动势 $\varepsilon = IR + Ir$.

其中，路端电压 $U = IR = \varepsilon - Ir = -r\cdot I + \varepsilon$，是 I 的线性函数.

电源总功率：$P_{总} = \varepsilon I = I^2R + I^2r$.
内阻功率：$P_{内} = I^2r$
电源输出功率：$P_{输} = I^2R$.

<1> $P_{输} = I^2R = (\varepsilon - Ir)I = -rI^2 + \varepsilon I = -r(I - \frac{\varepsilon}{2r})^2 + \frac{\varepsilon^2}{4r}$

当 $I = \frac{\varepsilon}{2r}$ 时，$P_{max} = \frac{\varepsilon^2}{4r}$

<2> $P_{输} = I^2R = (\frac{\varepsilon}{R+r})^2\cdot R = \frac{\varepsilon^2}{R + \frac{r^2}{R} + 2r} \leq \frac{\varepsilon^2}{4r}$.

此时 $R = \frac{r^2}{R}$，即 $r = R$.

<3> $P_{输} = UI = U\cdot\frac{\varepsilon - U}{r} = -\frac{1}{r}U^2 + \frac{\varepsilon}{r}U = -\frac{1}{r}(U - \frac{\varepsilon}{2})^2 + \frac{\varepsilon^2}{4r}$

当 $U = \frac{\varepsilon}{2}$ 时，$P_{max} = \frac{\varepsilon^2}{4r}$.

② 电流的微观解释：

$\vec{v_i} = \vec{v_{0i}} + \frac{e\vec{E}}{m}t_i$，而 $\vec{j} = ne\vec{v_i}$，$\therefore \vec{j} = ne\vec{v_{0i}} + \frac{ne^2\vec{E}}{m}t_i$

$\because \vec{v_{0i}}$ 为电子热运动产生的向各个方向的初速度 $\therefore n\vec{v_{0i}} = 0$

设电子平均自由飞行时间 $\tau = \frac{\sum t_i n_i}{n}$ $\therefore \vec{j} = \frac{ne^2\tau}{m}\vec{E}$ $\therefore \sigma = \frac{ne^2\tau}{m}$.

又∵ $\vec{v_i} = \vec{v_{0i}} + \frac{e\vec{E}}{m} t_i$

∴ $\frac{1}{2}mv_i^2 = \frac{e^2E^2}{2m}t_i^2 + \frac{1}{2}mv_{0i}^2 + \frac{et_i}{m}\vec{v_{0i}}\cdot\vec{E}$

对大量电子取平均：$\overline{\frac{1}{2}mv_i^2} = \overline{\frac{1}{2}mv_{0i}^2} + \frac{e^2E^2}{2m}\overline{t_i^2}$

∴ $\overline{\Delta E_k} = \overline{\frac{1}{2}mv_i^2} - \overline{\frac{1}{2}mv_{0i}^2} = \frac{e^2E^2}{2m}\overline{t_i^2}$

又∵ $\overline{t_i^2} = 2\tau^2$

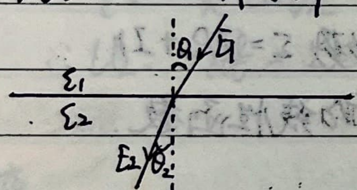
∴（单位体积中）热功率密度 $p = \frac{n\overline{\Delta E_k}}{\tau} = \frac{ne^2\tau}{m}E^2 = \sigma E^2$

∴ 热功率 $P = p\cdot V = \sigma E^2 ls = \frac{j^2 ls}{\sigma} = \frac{I^2 l}{s\sigma} = I^2R$

故电流做功 $\begin{cases} \text{总功：} W = qU = UIt \\ \text{发热：} Q = Pt = I^2Rt \end{cases}$ $\begin{cases} \text{若为纯电阻电路，} W = Q \Rightarrow U = IR \\ \text{若为非纯…} \quad W' = UIt - I^2Rt \end{cases}$

4. 边界条件.

① 场强的边界条件.

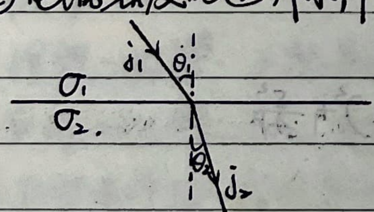


高斯定理：$\varepsilon_1 E_1\cos\theta_1 = \varepsilon_2 E_2\cos\theta_2$.

环路定理：$E_1\sin\theta_1 = E_2\sin\theta_2$.

∴ $\frac{tg\theta_1}{tg\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$

② 电流密度的边界条件：



稳恒条件：$j_1\cos\theta_1 = j_2\cos\theta_2$. 欧姆定律：$\vec{j} = \sigma\vec{E}$

环路定理：$\frac{j_1}{\sigma_1}\sin\theta_1 = \frac{j_2}{\sigma_2}\sin\theta_2$

∴ $\frac{tg\theta_1}{tg\theta_2} = \frac{\sigma_1}{\sigma_2}$

∴ 交界面处存自由电荷积累：$\sigma_e = \varepsilon_0\left(\frac{j_2}{\sigma_2}\cos\theta_2 - \frac{j_1}{\sigma_1}\cos\theta_1\right) = \varepsilon_0\frac{j_2}{\sigma_2}\frac{\sin(\theta_1-\theta_2)}{\sin\theta_1}$

5. 解复杂电路

① 基尔霍夫电流定律：

节点：$\sum_i I_i = 0$（节点可以是一片区域；电流流入为正流出为负或相反）

（理论依据：稳恒条件 $\oint \vec{j}\cdot d\vec{S} = -\frac{dq}{dt} = 0$.）

② 基尔霍夫电压定律.

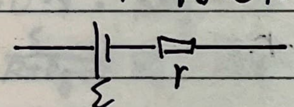
一段：$\sum_a^b U_i = \varphi_a - \varphi_b = \sum_i \varepsilon_i + \sum_i I_i R_i$ （电势下降为正上升为负或相反，电阻上电流方向为电势降落方向，电源上正极指向负极…）

回路：$\sum_L U_i = 0$

（理论依据：$U_{ab} = \varphi_a - \varphi_b = \int_a^b \vec{E} \cdot d\vec{l}$，$\oint_L \vec{E} \cdot d\vec{l} = 0$）

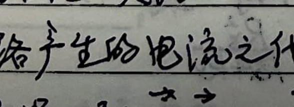
③ 戴维宁定理：

等效电压源 { ε = 网络开路端电压.
r = 除电源留内阻后网络的等效电阻.



④ 诺尔顿定理：

等效电流源 { I = 网络两端短接后端电流.
r = 除电源留内阻后网络的等效电阻.



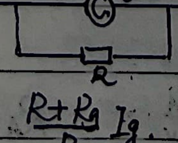
⑤ 叠加原理：

通过电路任一支路的电流等于各电源单独存在时（其它电源只留内阻）在该支路产生的电流之代数和.

（理论依据：$I = \vec{j} \cdot \vec{S} = \sigma \vec{E} \cdot \vec{S} = \sigma(\sum_i \vec{E}_i) \cdot \vec{S} = \sigma(\sum_i \frac{d\varepsilon_i}{dl} \hat{e}_i) \cdot \vec{S}$.）

6. 改装电流表 Ⓖ

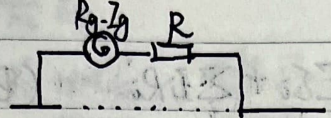
① 安培表 Ⓐ：并联分流小电阻.



量程：$I = \dfrac{I_g R_g}{\frac{R R_g}{R + R_g}} = \dfrac{R + R_g}{R} I_g$.

$\therefore R = \dfrac{R_g}{n-1}$.

② 伏特表 Ⓥ

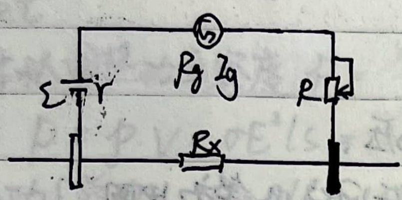


串联分压大电阻.

量程: $U=(R_g+R)I_g$

∴ $R=(n-1)R_g$

③ 欧姆表:



$$I_{测}=\frac{\varepsilon}{r+R_g+R+R_x} \Rightarrow R_x=\frac{\varepsilon}{I_{测}}-R-R_g-r.$$

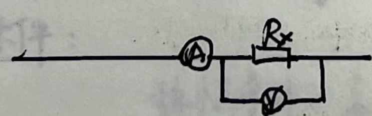
∵ 电流表刻度均匀,

∴ 安培表、伏特表刻度均匀, 欧姆表刻度不均匀.

7. 测量电阻、电流表内阻.

① 伏安法:

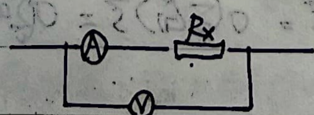
1) Ⓐ外接:



由于Ⓥ分流, $I_{测}>I_{实}$, 又∵ $R_{实}=\frac{U}{I_{实}}$, ∴ $R_{测}=\frac{U}{I_{测}}<R_{实}$.

$$R_{误}=R_{实}-R_{测}=R_x-\frac{U}{I_{测}}=R_x-\frac{R_xR_V}{R_x+R_V}=\frac{R_x^2}{R_x+R_V}.$$

2) Ⓐ内接:



由于Ⓐ分压, $U_{测}>U_{实}$, 又∵ $R_{实}=\frac{U_{实}}{I}$, ∴ $R_{测}=\frac{U_{测}}{I}>R_{实}$

$$R_{误}=R_{测}-R_{实}=\frac{U_{测}}{I}-R_x=(R_x+R_A)-R_x=R_A.$$

3) 选择Ⓐ的接法:

令 $\frac{R_x^2}{R_x+R_V}=R_A$, 则 $\frac{R_x}{R_V}=\frac{R_A}{R_x}+\frac{R_A}{R_V}$ ∵ $R_A\ll R_V$, ∴ $\frac{R_x}{R_V}\approx\frac{R_A}{R_x}$.

<1> 当 $R_x>\sqrt{R_AR_V}$ 时, 用内接法, 使电流表分压少.

<2> 当 $R_x<\sqrt{R_AR_V}$ 时, 用外接法, 使电压表分流少.

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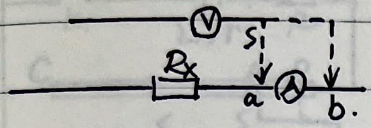
(3) 当R_x未知时：



让S与a、b分别接触一下

若电流表示数显著变化，S应接b.
(说明接a时Ⓥ分流不可，故R_x是大电阻，用内接法)
若电压表示数显著变化，S应接a.
(说明接b时Ⓐ分压不可，故R_x是小电阻，用外接法)

② 欧姆法：

$R_{测}=\frac{\varepsilon}{I_{测}}-R-R_g-r$

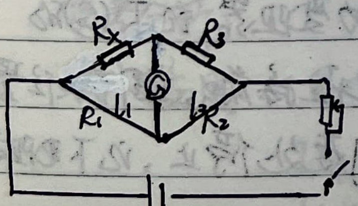
$I_{测}\to 0$时，$R_{测}\to\infty$.

$I_{测}=\frac{\varepsilon}{R+R_g+r}$时，$R_{测}=0$.

$\therefore R_{中}=\frac{\varepsilon}{\frac{(I_{测})_m}{2}}-R-R_g-r=R+R_g+r$

$\therefore R_{测}=\frac{\varepsilon}{I_{测}}-R_{中}$　. $R_{中}$为欧姆表中最中间刻度处的欧姆值，可直接读出.

③ 电桥法：



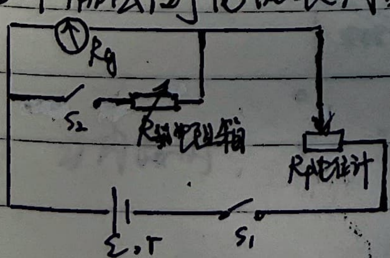
Ⓖ中$I=0$时，电桥平衡.

此时 $U_x=U_1$，$U_2=U_3$，$I_1=I_2$，$I_x=I_3$

$\therefore R_xI_x=R_1I_1$，$R_2I_2=R_3I_3$

$\therefore R_x=\frac{R_1}{R_2}\cdot R_3=\frac{L_1}{L_2}\cdot R_3$

④ 半偏法测电流表内阻.



闭合S_1，调节R_p使电流表满偏.

再闭合S_2，调节$R_{箱}$使电流表半偏，设此时$R_{箱}$电流为I'.

$\because \frac{I_g}{2}R_g=I'R_{箱}$　$\therefore I'=\frac{R_g}{2R_{箱}}\cdot I_g$

$\because$ 路端电压近似不变

$\therefore I_gR_g+I_gR_p=\frac{I_g}{2}R_g+(\frac{I_g}{2}+I')R_p=\frac{I_g}{2}R_g+(\frac{I_g}{2}+\frac{R_g}{2R_{箱}}I_g)R_p$

化简得：$R_g=\frac{R_{箱}}{1-\frac{R_{箱}}{R_p}}=\frac{R_pR_{箱}}{R_p-R_{箱}}$

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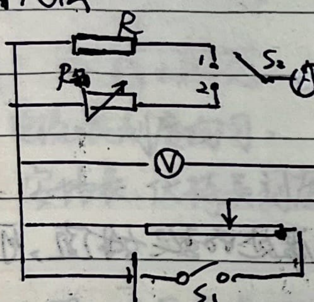
误差分析：$\begin{cases} I_g R_g + I_g(R_p + r) = \varepsilon \\ \frac{1}{2}I_g R_g + (\frac{1}{2}I_g + I')(R_p + r) = \varepsilon \\ \frac{1}{2}I_g R_g = I' R_{箱} \end{cases}$

$\therefore \dfrac{\varepsilon - I_g R_g}{\varepsilon - \frac{1}{2}I_g R_g} = \dfrac{I_g}{\frac{1}{2}I_g + I'} = \dfrac{I_g}{\frac{1}{2}I_g + \frac{R_g}{2R_{箱}} \cdot I_g} = \dfrac{2}{1 + \frac{R_g}{R_{箱}}}$

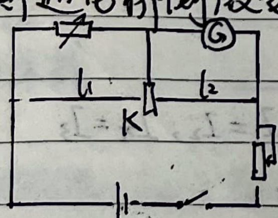
$\therefore \dfrac{R_g}{R_{箱}} = \dfrac{\varepsilon}{\varepsilon - I_g R_g}$ ——相对误差.

⑤ 替代法



将 S_2 打至1，调节滑动变阻器，记下Ⓐ的读数；再将 S_2 打至2，调节电阻箱阻值，使Ⓐ的读数与刚才一致，则 $R = R_g$.

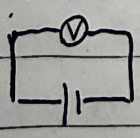
⑥. 惠斯通电桥测微安表内阻.



电路接通时，调节滑动变阻器，记下Ⓖ的读数；再将K与米尺电阻接触，调节其位置，在使Ⓖ与原先不变的刻度处停止，记下电阻箱阻值 $R_{箱}$，$\therefore R_g = \frac{l_2}{l_1} R_{箱}$.

8. 测量电动势：

① 伏特表法：



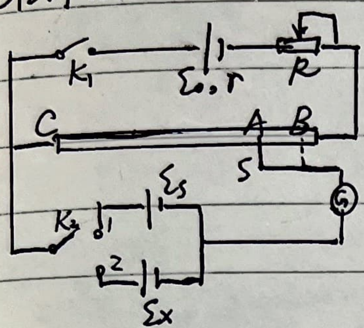
$\varepsilon = U$.

误差分析：$\varepsilon_{测} = U_{测}$，$\varepsilon_{实} = Ir + U_{测} = \dfrac{\varepsilon_{实} r}{R_V + r} + U_{测}$

$\therefore \varepsilon_{实} = \dfrac{R_V + r}{R_V} \cdot U_{测}$，测量值偏小.

$= \dfrac{R_V + r}{R_V} \cdot \varepsilon_{测}$.

② 电位计法：



闭合K_1；闭合K_2至1.

调节S至A，调节R使通过ε_s的电流为0.

$\therefore I = \dfrac{\varepsilon_0}{R_{AC}+R+r}$

$\therefore \varepsilon_s = I R_{AC} = \dfrac{\varepsilon_0 R_{AC}}{R_{AC}+R+r}$.

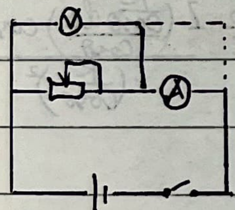
闭合K_2至2，将导线移至B使通过ε_x的电流为0.

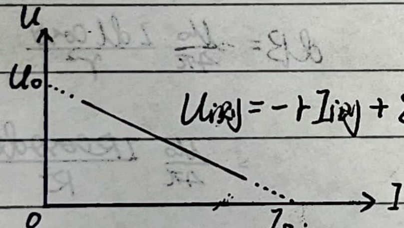
$\therefore I' = \dfrac{\varepsilon_0}{R_{BC}+R+r}$

$\therefore \varepsilon_x = I' R_{BC} = \dfrac{\varepsilon_0 R_{BC}}{R_{AC}+R+r}$

$\therefore \varepsilon_x = \dfrac{R_{BC}}{R_{AC}} \cdot \varepsilon_s = \dfrac{BC}{AC} \cdot \varepsilon_s$.

③ U-I图法：





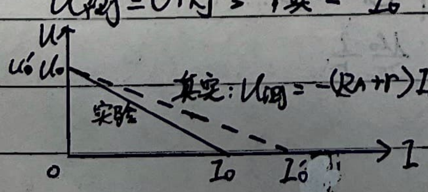
$U_{测} = -rI_{测} + \varepsilon$ ←实验后果计算 $\begin{cases} I_{测}=0，\varepsilon = U_0，断路 \\ U_{测}=0，r=\dfrac{U_0}{I_0}，短路 \end{cases}$

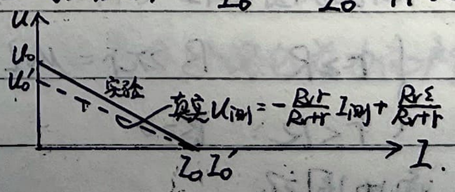
误差分析：

1)外接时. $\varepsilon = I_{测}(r+R_A) + U_{测}$，$\therefore U_{测} = -(R_A+r)I_{测} + \varepsilon$.

$I_{测}=0$时，$\varepsilon_{实} = U_0'$ 真实，

$U_{测}=0$时，$r_{实} = \dfrac{U_0'}{I_0'} - R_A$，故实验值 $r = \dfrac{U_0}{I_0} = \dfrac{U_0'}{I_0}$ 偏大，如图.





2)内接时，$\varepsilon = I_A r + U_{测} = \dfrac{U_{测}}{\dfrac{R_V(R_A+R)}{R_V+R_A+R}} \cdot r + U_{测}$

$= \dfrac{R_V+r}{R_V} U_{测} + I_{测} \cdot r$ $\therefore U_{测} = -\dfrac{R_V r}{R_V+r} I_{测} + \dfrac{R_V \varepsilon}{R_V+r}$

$I_{测}=0$时，$\varepsilon_{实} = \dfrac{R_V+r}{R_V} U_0'$，故实验值 $\varepsilon = U_0'$ 偏小.

$U_{测}=0$时，$r_{实} = \dfrac{\varepsilon}{I_0}$，故实验值 $r = \dfrac{U_0'}{I_0} = \dfrac{\varepsilon}{I_0}$ 真实，偏大

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十一.静磁场

1. 毕-萨定律:

$$d\vec{B}=\frac{\mu_0}{4\pi}\cdot\frac{I\,d\vec{l}\times\hat{r}}{r^2}\qquad \vec{B}=\int d\vec{B}$$

2.
高斯定理: $\oint_S \vec{B}\,d\vec{S}=0$

环路定理: $\oint_L \vec{B}\,d\vec{l}=\mu_0\sum I$

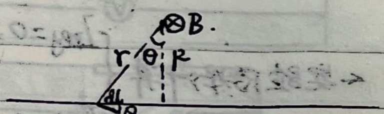
3. 典型载流体磁场:

① 无限长导线(电流I)

$\because B\cdot 2\pi r=\mu_0 I$

$\therefore B=\frac{\mu_0}{2\pi}\frac{I}{r}$

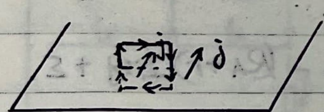
有限长导线(电流I).



$$dB=\frac{\mu_0}{4\pi}\frac{I\,dl\cos\theta}{r^2}=\frac{\mu_0}{4\pi}\frac{I\cdot\left(\frac{R}{\cos\theta}\frac{d\theta}{\cos\theta}\right)\cos\theta}{\left(\frac{R}{\cos\theta}\right)^2}$$

$$=\frac{\mu_0}{4\pi}\frac{IR\cos\theta\,d\theta}{R^2}$$

② 无限宽平板(电流密度j)



$\because 2BL=\mu_0 jL$

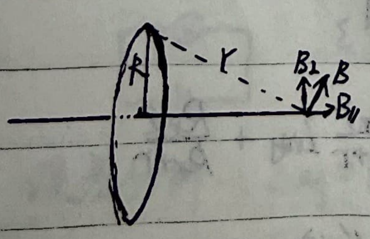
$\therefore B=\frac{1}{2}\mu_0 j$

③ 无限长圆柱面(半径R,表面均匀分布电流I).

$r\geq R$: $B\cdot 2\pi r=\mu_0 I$, $\therefore B=\frac{\mu_0}{2\pi}\frac{I}{r}$.

$r<R$: $B=0$.

④ 通电圆环.

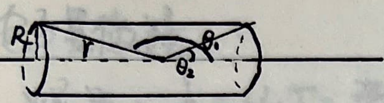


$\because dB=\frac{\mu_0}{4\pi}\frac{I\cdot dl}{r^2}=\frac{\mu_0}{4\pi}\frac{I\cdot R\,d\theta}{r^2}$

$\therefore B_\perp=0$, $B_{/\!/}=\frac{\mu_0}{4\pi}\left(\frac{I\,2\pi R}{r^2}\times\frac{R}{r}\right)=\frac{\mu_0 R^2 I}{2r^3}$

当 $r=R$ 时, $B=\frac{\mu_0 I}{2R}$.

⑤ 通电螺线管(单位长度匝数n,总匝数N,每匝电流I)



$\because dI = nI\,dl$, $\therefore dB = \frac{\mu_0 R^2 \cdot nI\,dl}{2r^3}$

又$\because dl = \left(\frac{R}{\sin\theta}d\theta\right)/\sin\theta = \frac{R}{\sin^2\theta}\cdot d\theta$，且 $r = \frac{R}{\sin\theta}$

$\therefore dB = \frac{\mu_0 nI}{2}\sin\theta\,d\theta$　　$\therefore B = \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)$

当管无限长时，$B = \mu_0 nI$.

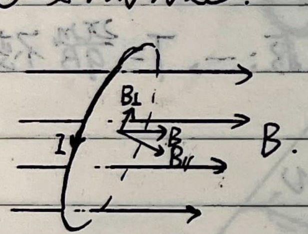
4. 洛仑兹力.

$\vec{F} = q\vec{v}\times\vec{B}$

对于一段导线元，$d\vec{F} = dq\,\vec{v}\times\vec{B} = nSe\,dl\,\vec{v}\times\vec{B} = nSev\,d\vec{l}\times\vec{B} = I\,d\vec{l}\times\vec{B}$

$\therefore \vec{F} = \int I\,d\vec{l}\times\vec{B}$ —安培力.

① 安培力力矩.



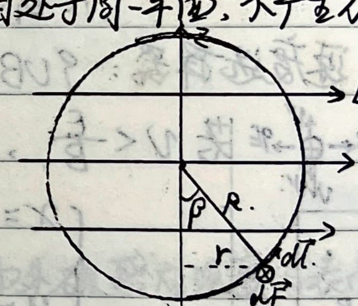
$dF_\perp = I\,dl\,B_\parallel$　$\because$与线圈处于同一平面，不产生力矩.

$dF_\parallel = I\,dl\,B_\perp\sin\beta$

$\therefore dM = dF_\parallel \cdot r$

$= I\,dl\,B_\perp\sin\beta\cdot r$

$= IR\,d\beta\,B_\perp\sin\beta\,R\sin\beta$



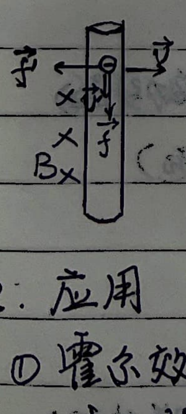
$\therefore M = IR^2B_\perp\int_0^{2\pi}\sin^2\beta\,d\beta = \pi\cdot IR^2B_\perp$　写成矢量形式 $\vec{M} = I\vec{S}\times\vec{B}$.

定义磁矩 $\vec{m} = I\vec{S}$　$\therefore \vec{M} = \vec{m}\times\vec{B}$.

② 洛仑兹力不做功

$P = \vec{F}\cdot\vec{v} = (q\vec{v}\times\vec{B})\cdot\vec{v}$

由于电子随导线在磁场中以 $\vec{v}$ 运动，电子受到沿导线的洛仑兹力而产生速度 $\vec{v'}$.



$P = (\vec{f'} + \vec{f})(\vec{v} + \vec{v'})$

$= \vec{f}\cdot\vec{v'} + \vec{f'}\cdot\vec{v}$

$= evB\cdot v' - ev'Bv$

$= 0$

3. 应用

① 霍尔效应.

$\because E_H = vB$，$U_H = E_H h = vBh$. 又$\because I = nbh\cdot qv$. $\therefore U_H = \frac{IB}{nqb}$

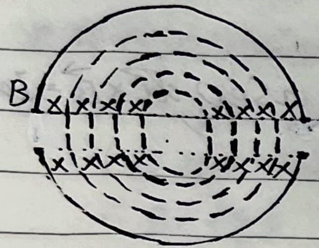
故霍尔电阻 $R_H = \frac{B}{nqb}$.

② 电流表：$BIS = k\theta$.

③ 质谱仪：$m\frac{v^2}{r} = qvB$

∴ $r = \frac{mv}{qB}$，$T = \frac{2\pi m}{qB}$.

④ 回旋加速器.



$qU = \frac{1}{2}mv_1^2$，$r_1 = \frac{mv_1}{qB}$，$v_1 = \sqrt{\frac{2qU}{m}}$

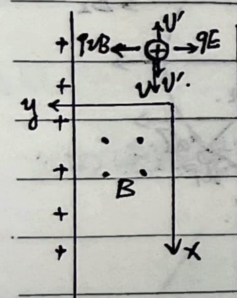
$qU = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$，$r_2 = \frac{mv_2}{qB}$，$v_2 = \sqrt{\frac{4qU}{m}}$

$qU = \frac{1}{2}mv_3^2 - \frac{1}{2}mv_2^2$，$r_3 = \frac{mv_3}{qB}$，$v_3 = \sqrt{\frac{6qU}{m}}$

....

∴ $v_1 : v_2 : v_3 : \cdots = r_1 : r_2 : r_3 : \cdots = 1 : \sqrt{2} : \sqrt{3} : \cdots$，$T = \frac{2\pi m}{qB}$ 不变.

⑤ 速度选择器：$qvB = qE$. ∴ $v = \frac{E}{B}$.



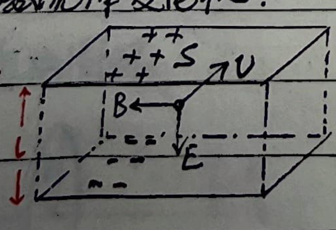
若 $v < \frac{E}{B}$，$qB(v+v') = qE$，∴ $v' = \frac{E}{B} - v$.

$$\begin{cases} x = \frac{E}{B}t - \frac{m(\frac{E}{B}-v)}{qB}\sin(\frac{qB}{m}t) \\ y = \frac{m(\frac{E}{B}-v)}{qB}\cos(\frac{qB}{m}t) - \frac{m(\frac{E}{B}-v)}{qB} \end{cases}$$

若 $v > \frac{E}{B}$，$qBv = qE + qBv'$，∴ $v' = v - \frac{E}{B}$

$$\begin{cases} x = \frac{E}{B}t + \frac{m(v-\frac{E}{B})}{qB}\sin(\frac{qB}{m}t) \\ y = \frac{m(v-\frac{E}{B})}{qB} - \frac{m(v-\frac{E}{B})}{qB}\cos(\frac{qB}{m}t) \end{cases}$$

⑥ 磁流体发电机.



$\begin{cases} \text{非静电力：} E_{ne} = \frac{f}{q} = vB \ \text{（由洛伦兹力引起）} \\ \text{静电力：} E_s \ \text{（由上下电荷积累引起）} \end{cases}$

∴ $I = \sigma(E_{ne} - E_s)\cdot S$ （σ：电导率）

总功率：$P = IE_{ne}l$

内阻功率：$P = I(E_{ne} - E_s)l$.

输出功率：$P = IE_sl = \sigma(vB - E_s)E_sV \leq \frac{1}{4}\sigma v^2B^2V$ （V为电机间总体积）.

⑦ 电子显微镜：

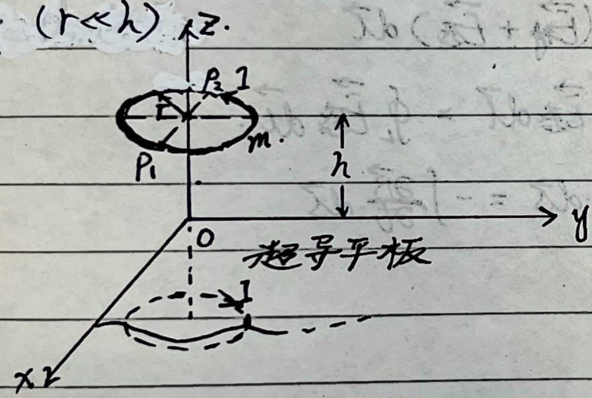
螺距：$h = v_{\parallel} \cdot T = \frac{2\pi m}{qB} \cdot v_{\parallel} \approx \frac{2\pi m}{qB} \cdot v$.

⑧ 磁镜：

$$|\vec{m}| = |\vec{S} \cdot I| = \frac{\pi R^2 q}{T} = \frac{\pi \frac{m^2 v^2}{q^2 B^2} \cdot q}{\frac{2\pi m}{qB}} = \frac{\frac{1}{2}mv^2}{B}.$$

由于磁矩守恒. $\frac{mv_\perp^2}{2B_0} = \frac{mv^2\sin^2\theta}{2B_0} = \frac{mv^2}{2B}$ $\quad\therefore \sin\theta_m = \sqrt{\frac{B_0}{B_m}}$

⑨ 像电流：$(r \ll h)$



由于 $r \ll h$，且超导体内部磁感应强度为零，外表面外磁感应强度与表面平行，故等效于一个 $z=-h$ 处的环状像电流，两环状电流反向平行.

$$mg = \frac{\mu_0}{2\pi}\frac{I}{2h} \cdot 2\pi r \cdot I$$

$$\therefore I = \sqrt{\frac{2mgh}{\mu_0 r}}$$

1) 若圆环上下微振动，$F = \frac{\mu_0 r I^2}{2(h-x)} - mg \approx -mg + \frac{\mu_0 r I^2}{2h}\left(1+\frac{x}{h}\right) = \frac{\mu_0 r I^2}{2h^2}x$

$$\therefore T = 2\pi\sqrt{\frac{2mh^2}{\mu_0 r I^2}} = 2\pi\sqrt{\frac{h}{g}}$$

2) 若以 P_1P_2 为轴微摆动，$M \approx \frac{\mu_0 r I^2}{2}\left(\frac{1}{2h - r\cdot\theta} - \frac{1}{2h + r\cdot\theta}\right) \cdot r \approx \frac{\mu_0 I^2 r^2}{4h^2} \cdot \theta$

$= \frac{mgr^2}{2h}\theta$，$\therefore T = 2\pi\sqrt{\frac{\frac{1}{2}mr^2}{\frac{mgr^2}{2h}}} = 2\pi\sqrt{\frac{h}{g}}$.

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十二、电磁感应与交流电.

1. 法拉第电磁感应定律.

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B}\, d\vec{S}.$$

ε方向由右手螺旋定则应用于环流方向上确定；

若Φ随时间变大，则ε与环流方向相反；反之，则与环流同向.

①动生：$|\varepsilon| = \frac{d\Phi}{dt} = \frac{d}{dt}(Blx) = Blv$

对于一般导体：$\varepsilon_{ab} = \int_a^b (\vec{v}\times\vec{B})\cdot d\vec{l}$.

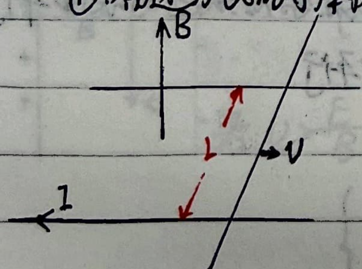
②感生：$|\varepsilon| = \oint_L \vec{E}\, d\vec{l} = \oint_L(\vec{E}_{静} + \vec{E}_{感})\, d\vec{l}$

$$= \oint_L \vec{E}_{静}\cdot d\vec{l} + \oint_L \vec{E}_{感}\, d\vec{l} = \oint_L \vec{E}_{感}\, d\vec{l}.$$

$$\therefore \oint_L \vec{E}_{感}\, d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

2. 典例：

①初速为v_0的导体棒.



$\because F = BIL$

动量：$m\frac{dv}{dt} = BL\cdot\frac{BLv}{R} = \frac{B^2L^2v}{R}$

$\therefore \frac{dv}{v} = \frac{B^2L^2}{mR}dt$

$\therefore \ln\frac{v_0}{v} = \frac{B^2L^2}{mR}t$

$\therefore v = v_0 e^{-\frac{B^2L^2}{mR}t}$

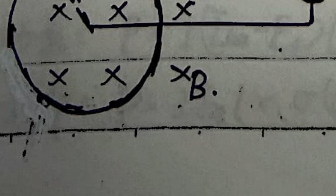
能量：$dW = I\varepsilon dt = IBlv\, dt$

而 $Fdt = BIl\, dt = m\, dv$

$\therefore dW = IBlv\frac{m\, dv}{BIL} = mv\, dv$

$\therefore \int I\varepsilon\, dt = \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$. 能量守恒.

② 旋转圆盘.



$$\varepsilon = \frac{B\, dS}{dt} = \frac{B\cdot\frac{1}{2}d\theta\cdot R^2}{dt} = \frac{1}{2}B\omega R^2.$$

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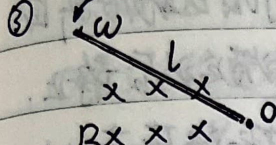
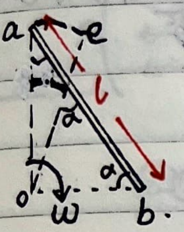
③



$$\varepsilon = \int Bv\,dl = \int B\omega l\,dl = \frac{1}{2}B\omega l^2.$$



$$\varepsilon_{ab} = \varepsilon_{ac} = U_a - U_b = \frac{1}{2}B\omega(l\cos\alpha)^2 - \frac{1}{2}B\omega(l\sin\alpha)^2$$
$$= \frac{1}{2}B\omega l^2\cos 2\alpha.$$

④圆环导线的感生电动势.

$$\oint_L \vec{E}\cdot d\vec{l} = E\cdot 2\pi r$$

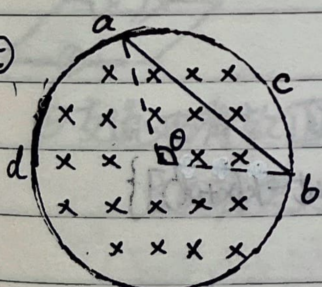
$r>R$时, $E\cdot 2\pi r = \pi R^2\frac{dB}{dt}$

$\therefore E = \frac{R^2}{2r}\frac{dB}{dt}$

$r<R$时, $E\cdot 2\pi r = \pi r^2\frac{dB}{dt}$

$\therefore E = \frac{r}{2}\frac{dB}{dt}.$

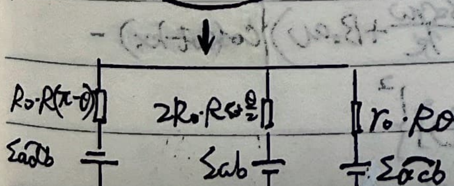
⑤



$$\varepsilon_{\widehat{acb}} = \frac{1}{2}\theta R^2\cdot\frac{dB}{dt}$$

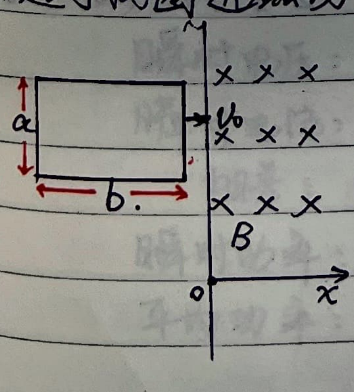
平面abc中, $\varepsilon = \frac{d\Phi}{dt} = (\frac{1}{2}\theta R^2 - \frac{1}{2}\sin\theta R^2)\frac{dB}{dt}.$

$$\therefore \varepsilon_{ab} = \varepsilon_{\widehat{acb}} - \varepsilon = \frac{1}{2}\theta R^2\frac{dB}{dt} - \frac{1}{2}\theta R^2\frac{dB}{dt} + \frac{1}{2}\sin\theta R^2\frac{dB}{dt}$$
$$= \frac{1}{2}\sin\theta R^2\frac{dB}{dt}.$$



其中: ε为环路acba的总电动势.

⑥超导线圈进磁场.(自感L)



磁通量 $\Phi = Bax = Li$

而 $F = -Bai = m\frac{d^2x}{dt^2}$

$\therefore Bax + \frac{mL}{Ba}\frac{d^2x}{dt^2} = 0.$ 即 $\frac{B^2a^2}{L}x + m\frac{d^2x}{dt^2} = 0.$

$\therefore x = A\sin(\sqrt{\frac{B^2a^2}{mL}}t)$ $\therefore v = \frac{dx}{dt} = A\sqrt{\frac{B^2a^2}{mL}}\cos\sqrt{\frac{B^2a^2}{mL}}t.$

当$t=0$时, $v = A\sqrt{\frac{B^2a^2}{mL}} = v_0$ $\therefore A = v_0\sqrt{\frac{mL}{B^2a^2}} = \frac{v_0}{Ba}\sqrt{mL}.$

$\therefore x = \frac{v_0\sqrt{mL}}{Ba}\sin(\frac{Ba}{\sqrt{mL}}t).$ $v = v_0\cos(\frac{Ba}{\sqrt{mL}}t)$

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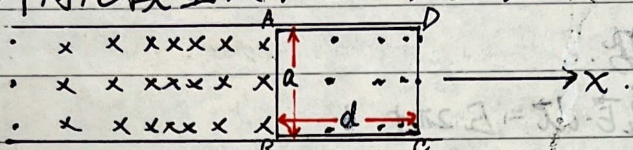
1) 当 $A=\frac{v_0\sqrt{mL}}{Ba}<b$，即 $v_0<\frac{Bab}{\sqrt{mL}}$ 时，线圈做半个周期的简谐运动后折回；

2) 当 $A=\frac{v_0\sqrt{mL}}{Ba}=b$，即 $v_0=\frac{Bab}{\sqrt{mL}}$ 时，线圈做完半个周期简谐运动后静止.

3) 当 $A=\frac{v_0\sqrt{mL}}{Ba}>b$，即 $v_0>\frac{Bab}{\sqrt{mL}}$ 时，线圈未完成半周期谐振动后就以匀速直进. $\frac{1}{2}mv_0^2=\frac{1}{2}\frac{B^2a^2}{L}\cdot b^2+\frac{1}{2}mv^2$，$v=\sqrt{v_0^2-\frac{B^2a^2b^2}{mL}}$

⑦ 磁悬浮列车简化模型.（轨道磁场 $B=B_0\cos(\omega t-kx)$）



设此时线框速度为 v、AB边坐标为 x，时刻为 t.

$$\therefore \varepsilon_{感}=-\int_S\frac{\partial B}{\partial t}dS=\omega B_0\int_x^{x+d}\sin(\omega t-kx)\cdot a\,dx$$

$$=\frac{-\omega B_0 a}{k}\int_x^{x+d}\sin(\omega t-kx)\,d(\omega t-kx)$$

$$=\frac{\omega B_0 a}{k}\{\cos[\omega t-k(x+d)]-\cos(\omega t-kx)\}.$$

$$\varepsilon_{动}=[B(x,t)-B(x+d,t)]av$$

$$=B_0av\{\cos(\omega t-kx)-\cos[\omega t-k(x+d)]\}$$

$$\therefore \varepsilon_{总}=\varepsilon_{动}+\varepsilon_{感}=\left(-\frac{B_0a\omega}{k}+B_0av\right)\{\cos(\omega t-kx)-\cos[\omega t-k(x+d)]\}$$

$$\therefore F_{总}=[B(x+d,t)-B(x,t)]I\cdot a$$

$$=-B_0\{-\cos[\omega t-k(x+d)]+\cos(\omega t-kx)\}^2\left(-\frac{B_0a\omega}{k}+B_0av\right)$$

$$=\frac{B_0^2a^2(\frac{\omega}{k}-v)}{R}\{\cos(\omega t-kx)-\cos[\omega t-k(x+d)]\}^2$$

和差化积，得 $F=\left[\frac{4B_0^2a^2(\frac{\omega}{k}-v)}{R}\sin^2(\frac{kd}{2})\right]\sin^2\left[(\omega t-kx)-\frac{kd}{2}\right]=F_0\sin^2(\omega t-kx-\frac{kd}{2})$

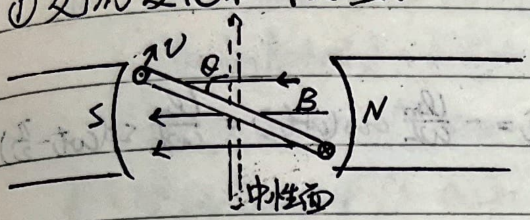
1) 当 $kd=2n\pi$ 时，$(F_0)_{min}=0$

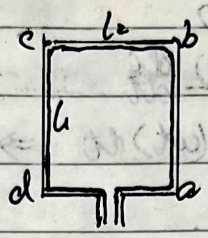
2) 当 $kd=(2n+1)\pi$ 时，$(F_0)_{max}=\frac{4B_0^2a^2(\frac{\omega}{k}-v)}{R}$

3) 当 d 取其它值时. $F_0\in\left(0,\frac{4B_0^2a^2(\frac{\omega}{k}-v)}{R}\right)$

3. 交流电.

①交流发电机模型.





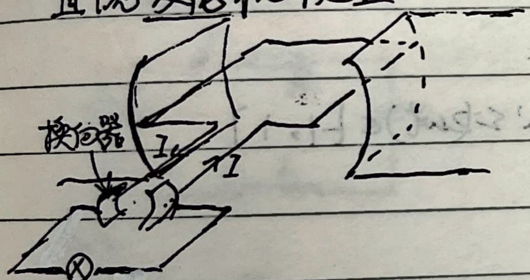
{旋转电枢(线圈)式
旋转磁极式.

$e_{总} = e_{ab}+e_{cd} = 2BL_1 v\sin(\omega t)$

$= 2BL_1\cdot\omega\frac{L_2}{2}\sin(\omega t) = BL_1L_2\omega\sin(\omega t) = BS\omega\sin(\omega t) = \phi_m\omega\sin(\omega t) = \varepsilon_m\sin(\omega t)$

若有n匝线圈 $e = 2nBL_1 v\sin(\omega t)$.

直流发电机模型:



对于直流电动机:

$I = \frac{U-\varepsilon}{R}$ (ε为反电动势)

∴ $U = \varepsilon + IR$ (U为电源电压)

∴ $UI = \varepsilon I$(机械能) $+ I^2R$(热能).

② 交流电有效值: 一个周期内有效电流(直流恒定)发热量与实际相当.

根据定义: $I_{效}^2RT = R\int_0^T i^2dt = RI_m^2\int_0^T\sin^2(\omega t)dt$

∴ $I_{效}^2T = \frac{I_m^2}{\omega}\int_0^{\omega T}\sin^2(\omega t)d(\omega t) = \frac{I_m^2}{2\omega}[\omega T - \frac{1}{2}\sin(2\omega T)]$

∵ $T=\frac{2\pi}{\omega}$ ∴ $\sin(2\omega T)=\sin 4\pi=0$

∴ $I_{效}^2T = \frac{I_m^2}{2\omega}\cdot\omega T = \frac{1}{2}I_m^2 T$

∴ $I_{效} = \frac{I_m}{\sqrt{2}}$.

③ 纯电阻电路:

瞬时电压: $u = U_m\sin(\omega t)$.

瞬时电流: $i = \frac{U_m}{R}\sin(\omega t)$.

相差: u与i同相.

瞬时功率: $P = ui = U_mI_m\sin^2(\omega t) = \frac{U_mI_m}{2}[1-\cos(2\omega t)] = U_{效}I_{效}[1-\cos(2\omega t)]$

平均功率: $\bar{P} = U_{效}I_{效}[1-\cos(2\omega t)]$ ∵ $\cos(2\omega t)\in[-1,1]$

∴ $\bar{P} = U_{效}I_{效}$.

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④ 纯电感电路：

瞬时电压：$u_L = U_m \sin(\omega t)$.

瞬时电流：$\because u_L = -\varepsilon_L = L\frac{di}{dt}$

$\therefore di = \frac{U_m}{L}\sin(\omega t)dt \Rightarrow i = -\frac{U_m}{\omega L}\cos(\omega t) = \frac{U_m}{\omega L}\sin(\omega t - \frac{\pi}{2})$

相差：u_L比i超前$\frac{\pi}{2}$.

感抗：令$X_L = \omega L$，则$i = \frac{U_m}{X_L}\sin(\omega t - \frac{\pi}{2})$

即$X_L = 2\pi f L$.

瞬时功率：$P = u_L i = -U_m I_m \sin(\omega t)\cos(\omega t)$

$= -\frac{1}{2}U_m I_m \sin(2\omega t)$

$= -U_{有效}I_{有效}\sin(2\omega t)$.

平均功率：$\overline{P} = -U_{有效}I_{有效}\overline{\sin(2\omega t)}$　　$\because \sin(2\omega t) \in [-1, 1]$

$\therefore \overline{P} = 0$.

⑤ 纯电容电路：

瞬时电压：$u_C = U_m \sin(\omega t)$.

瞬时电流：$\because dq = C du_C$

$\therefore \frac{dq}{dt} = i = C\frac{du_C}{dt} = C\omega U_m \cos(\omega t) = \frac{U_m}{\frac{1}{C\omega}}\sin(\omega t + \frac{\pi}{2})$

相差：i比u_C超前$\frac{\pi}{2}$.

容抗：令$X_C = \frac{1}{C\omega}$，则$i = \frac{U_m}{X_C}\sin(\omega t + \frac{\pi}{2})$

即$X_C = \frac{1}{2\pi f C}$

瞬时功率：$P = u_C i = U_m I_m \sin(\omega t)\cos(\omega t)$

$= \frac{1}{2}U_m I_m \sin(2\omega t)$

$= U_{有效}I_{有效}\sin(2\omega t)$

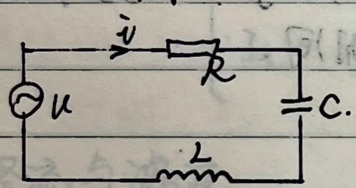
平均功率：$\overline{P} = \frac{\int_0^T P\,dt}{T} = 0$.

⑥ 交流电功率：(设 u比i超前 φ)

$$p = ui = U_m \sin(\omega t)\cdot I_m \sin(\omega t-\varphi)$$
$$= U_m I_m[\sin(\omega t)\cdot\sin(\omega t)\cos\varphi - \sin(\omega t)\sin\varphi\cos(\omega t)]$$
$$= U_m I_m\cos\varphi\cdot\frac{1}{2}[1-\cos(2\omega t)] - U_m I_m \sin\varphi\cdot\frac{1}{2}\sin(2\omega t)$$
$$\overline{P} = \frac{1}{2} U_m I_m\cos\varphi\,\overline{[1-\cos(2\omega t)]} - \frac{1}{2} U_m I_m\sin\varphi\,\overline{\sin(2\omega t)}$$
$$= \frac{1}{2} U_m I_m\cos\varphi = U_{效} I_{效}\cos\varphi.$$

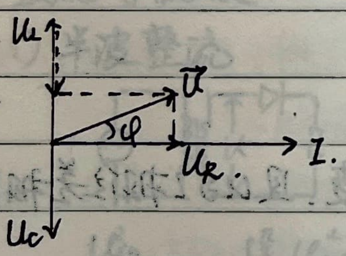
⑦ 阻抗(Z)

1) RCL串联：



由于串联，各元件电流矢量相同.

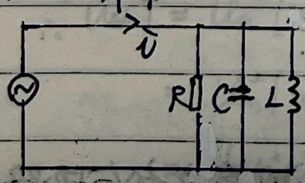
电压矢量满足 $\vec{U} = \vec{U_R}+\vec{U_C}+\vec{U_L}$，如图



$$|\vec{U}| = \sqrt{U_R^2+(U_L-U_C)^2}$$
$$=\sqrt{(IR)^2+(I\omega L - I\cdot\frac{1}{\omega C})^2}$$
$$= I\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}$$

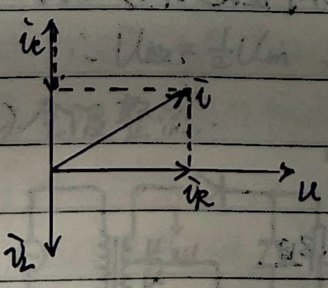
$\therefore Z = \frac{U}{I} = \sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}$ 其中，U、I皆为有效值.

2) RCL并联：



由于并联，各元件电压矢量相同.

电流矢量满足 $\vec{i}=\vec{i_R}+\vec{i_C}+\vec{i_L}$，如图.



$$|\vec{i}| = \sqrt{i_R^2+(i_C-i_L)^2}$$
$$=\sqrt{(\frac{U}{R})^2+(\frac{U}{\frac{1}{\omega C}}-\frac{U}{\omega L})^2}$$
$$= U\sqrt{\frac{1}{R^2}+(\frac{1}{\omega L}-\omega C)^2}$$

$\therefore Z = \frac{U}{I} = [\frac{1}{R^2}+(\frac{1}{\omega L}-\omega C)^2]^{-\frac{1}{2}}$ 其中U、I为有效值.

3) 串联谐振：

$$\varphi = \mathrm{arctg}\frac{U_L-U_C}{U_R} = \mathrm{arctg}\frac{\omega L-\frac{1}{\omega C}}{R}$$

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当 $\omega L-\frac{1}{\omega C}=0$，即 $\omega=\sqrt{\frac{1}{LC}}$ 时，发生串联谐振.

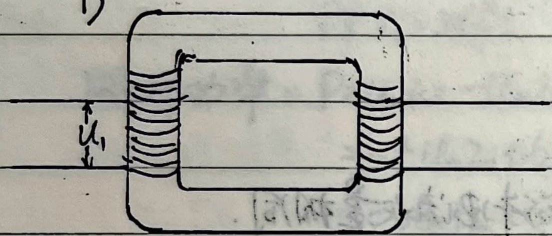
4) 并联谐振：

$$\varphi=\arctg\frac{I_C-I_L}{I_R}=\arctg\frac{\omega C-\frac{1}{\omega L}}{\frac{1}{R}}$$

当 $\omega C-\frac{1}{\omega L}=0$，即 $\omega=\sqrt{\frac{1}{LC}}$ 时，发生并联谐振.

⑧ 变压器.

1)



原线圈外加电压 $U_1=U_m\sin(\omega t)$，当 U_1 随时间作周期变化时，铁芯中相应产生变化的磁通量，由于铁芯完全导磁（理想）原副线圈中 $\frac{d\Phi}{dt}$ 相同.

原线圈中感应电动势 $\varepsilon_1=U_1-Ir_{原}$，$\varepsilon_2=U_2$

∵ $r_{原}$ 很小，∴ $\varepsilon_1\approx U_1$.

又∵ $\varepsilon_1=N_1\frac{d\Phi}{dt}$，$\varepsilon_2=N_2\frac{d\Phi}{dt}$，

∴ $\frac{\varepsilon_1}{\varepsilon_2}=\frac{N_1}{N_2}=\frac{U_1}{U_2}$.

又∵ 理想状态下，原副线圈中电流功率不变. 且U与I相位差相同.

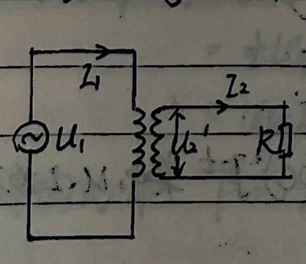
∴ $U_1I_1\cos\varphi_1=U_2I_2\cos\varphi_2$

∴ $\frac{I_1}{I_2}=\frac{U_2}{U_1}=\frac{N_2}{N_1}$

若有多个副线圈，由能量守恒：$U_1I_1=\sum_{i=2}^{n}U_iI_i$.

∵ $\frac{U_1}{N_1}=\frac{U_i}{N_i}$，∴ $N_1I_1=\sum_{i=2}^{n}N_iI_i$

由副线圈连接的部分，相当于一阻抗，可通过调整这部分电路的总阻抗，达到原线圈连接部分的阻抗匹配.



∵ $R'=\frac{U_1}{I_1}$，$R=\frac{U_2}{I_2}$

而 $\frac{U_1}{U_2}=\frac{N_1}{N_2}$，$\frac{I_1}{I_2}=\frac{N_2}{N_1}$

∴ $R'=\left(\frac{N_1}{N_2}\right)^2R$，即等效电阻.